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TOPICS IN THE THEORY OF SURFACES IN ELLIPTIC SPACE

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Theory of Surfaces
in Elliptic Space

by A. V. Pogorelov

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Preface

The theory of convex surfaces in Euclidean space is now so highly developed that a large proportion of the remaining open problems are questions of refinements of a somewhat technical nature. It is therefore not surprising that many investigators prefer generalizing the theory, for instance, by considering other spaces, to working on these technical refinements. Elliptic space offers one of the most interesting possibilities for further research. By elliptic space is meant three-dimensional projective space with a metric of constant curvature. It can be "visualized" as the unit sphere in four-dimensional Euclidean space with antipodal points identified.

Pogorelov's purpose here is to show how rigidity and regularity theorems for convex surfaces can be extended to surfaces in elliptic space. This could have been accomplished either by adapting arguments which work in Euclidean space to elliptic space, or by developing a more general theory which would apply to both spaces. However, instead of following either of these procedures, the author develops more interesting methods based on principles which enable one to transfer results directly from Euclidean space to elliptic space. These methods have the advantages of making a relatively short exposition possible and of being of some interest in themselves. Thus, what might have been a long and tedious book turns out to be a short and interesting one. (However, the chapter devoted to regularity is exceptional. In it, the author, for reasons which are not clear to us, evidently preferred to work entirely in elliptic space rather than to exploit his principles of transference to the fullest extent.)

One may admire the skill with which Pogorelov carries out his project and still question the interest of developing a theory of convex surfaces in elliptic space. Therefore, in order

to show that the results obtained here are more than empty exercises, we shall discuss a problem which is stated entirely in terms of Euclidean space, but which seems to be much easier to solve if one uses Pogorelov's principles of transference to convert it to a problem about surfaces in elliptic space.

J. J. Stoker has discussed the infinitesimal rigidity of complete open convex surfaces in Euclidean space.[*] He showed that certain cases of the theorem stated below are correct and conjectured that the theorem itself is true.

Theorem. Let Φ be a complete open surface with positive Gaussian curvature in Euclidean space. Then Φ does not admit nontrivial infinitesimal deformation $z = (z^1, z^2, z^3)$ such that $z = o\,|x|$ as $|x| \to \infty$, where x denotes a point on Φ.

Apparently this theorem has not been proved before. We shall sketch a very simple proof based on Pogorelov's methods.

Stoker proved that the hypotheses imply that, for a suitable choice of orthogonal coordinates (x^1, x^2, x^3) in Euclidean space, the surface Φ can be represented in the form $x^3 = f(x^1, x^2)$, where f is defined on a convex subset of the (x^1, x^2) plane. The map S, which sends the point x of Φ to the point $Sx = (x + e_0)/(1 + x^2)^{1/2}$ in elliptic space, maps Φ to a convex surface F in elliptic space (cf. Chapter II para. 2). The deformation vector z will be mapped to a deformation vector $\zeta = [z - e_0\,(xz)]/(1 + x^2)^{1/2}$ for F, and ζ will be trivial if and only if z is trivial (cf. Chapter V, para. 2).

Let F' be the set of point of the topological closure of F which are not in F itself. Then F' is a subset of the "equator" $y^0 = 0$ of the sphere S^4: $(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = 1$. $F \cup F'$ will be a convex surface with boundary which is "visible" from the point $P_0 = (0,0,0,1)$. Also, it is easy to see that $F \cup F'$ lies in an open hemisphere of S^4. It is not possible, in general, to extend ζ to a deformation vector which is continuous on $F \cup F'$. However, if V denotes the unit vector which is tangent at a point y of F to the great circle passing through y and P_0, then the property $z + o\,|x|$ is easily seen to imply that

*Courant Anniversary Volume, Interscience Publishers, Inc. (1948), pp. 407–422.

the inner product $V \zeta \to 0$ as $y \to y_0$, where y_0 is any point of F''. In other words, the distance from P_0 under the infinitesimal deformation ζ is stationary on F''. The proof is completed by applying the analogue for elliptic space of the theorem which states that a convex surface in Euclidean space is infinitesimally rigid with respect to deformations which leave the distances of boundary points from some fixed point stationary, provided the surface is "visible" from that point, and lies on one side of a plane through the point cf. ch. 5, para. 3. (The method of proof used by Alexsandrov, and Senkin [5] makes it clear that the fact that the deformation vector cannot be extended continuously to the boundary does not cause any difficulty.)

It would, no doubt, be possible to carry out a proof of the theorem without using elliptic space by employing suitable projective transformations. However, such a proof would probably seem rather formal and artificial, whereas the proof just sketched is very intuitive. This is just one of a number of examples which could have been given to illustrate the usefulness of a theory of surfaces in elliptic space.

This translation was prepared by Royer and Roger, Inc. and I have been responsible for the technical editing of it. In most cases, we have used the terminology established by Busemann in his book,* which we suggest as a source for the definitions of concepts which may be unfamiliar to the reader, and as an introduction to the topics needed for an understanding of this book. We have kept the author's designations for theorems and formulae. Thus, for example, the equations known as the Codazzi-Mainardi equations in the West are called the Peterson-Codazzi equations and the theorem usually attributed to Busemann and Feller is called Busemann's Theorem. We have made a serious effort to locate, correct, and clarify the few misprints, misstatements, and ambiguities which existed in the original. Usually, this has been done without comment, but in a few cases footnotes have been added. Finally, for the benefits of readers who do not know

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*Convex Surfaces, Interscience Publishers, Inc., (1958)

Russian, it should be noted that a number of the references have been published in German by the Akademie Verlag, Berlin.

August, 1961   RICHARD SACKSTEDER

‡Note added in proof: After this preface was written it was discovered that the proof which Aleksandrov and Senkin [5] give of their theorem is not valid in the limiting case used below, where F' lies in a plane through P_0. Thus we cannot claim here to give a proof of the theorem stated above, but rather to show its equivalence to the limiting case of Aleksandrov and Senkin's theorem. We intend to discuss this more fully in another place.

Introduction

This book deals with the solution of a number of problems in the theory of surfaces in elliptic space, through consideration of isometric surfaces. The principal method of investigation is comparison of a pair of isometric figures in elliptic space with a pair of isometric figures in a Euclidean space which corresponds geodesically to the elliptic space. This enables us to transpose the main difficulties in the proof to Euclidean space, where they can be overcome by the appropriate theorems.

The essence of this transformation of isometric figures in elliptic space is as follows.

Let R be an elliptic space with a curvature K = 1. Let us introduce the Weierstrass coordinates x_i (i = 0, 1, 2, 3) into R and compare each point in the space R with a pair of points in the four-dimensional Euclidean space with the Cartesian coordinates x_i and $-x_i$. These points will cover a unit sphere, since the Weierstrass coordinates satisfy the equation

$$x^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1.$$

Let us use E_0 to denote the three-dimensional Euclidean space $x_0 = 0$.

Let us assume we have two isometric figures F' and F'' in the Euclidean space R. Let x' be an arbitrary point in the figure F', and x'' be the isometrically

corresponding point in F". Then the equations

$$y = \frac{x' - e_0\,(x'e_0)}{e_0\,(x' + x'')}\,, \qquad y = \frac{x'' - e_0\,(x''e_0)}{e_0\,(x' + x'')}\,,$$

where e_0 is the unit vector along the axis x_0, give the two isometric figures Φ' and Φ'' in the Euclidean space E_0. If the figures F' and F" are congruent, the figures Φ' and Φ'' are also congruent. Conversely, if Φ' and Φ'' are congruent, F' and F" are congruent.

There is also a corresponding theorem on the transformation of isometric figures in Euclidean space into isometric figures in elliptic space.

If F' and F" are isometric convex surfaces, the convexity of the corresponding surfaces Φ' and Φ'' in Euclidean space (Chapter IV) is guaranteed whenever F' and F" have been moved to suitable positions. This enables us to extend various uniqueness theorems for surfaces in Euclidean space to surfaces in elliptic space (Chapter VI).

Since consideration of infinitesimal deformations of a surface involves infinitely close isometric surfaces, this transformation enables us to reduce the problem of infinitesimal deformations of surfaces in elliptic space to the problem of infinitesimal deformations of surfaces in Euclidean space and to prove the corresponding theorems (Chapter V).

In addition to infinitesimal rigidity and uniqueness, the book also considers the problem of the regularity of convex surfaces with a regular metric; this problem is solved in Chapter VII.

To make the book more comprehensible, the first two sections give a brief description of the fundamentals of the theory of curves and surfaces in elliptic space.

We should point out in conclusion that the results of the present work can be extended to Lobachevsky space without any great changes, and that the methods used may be applied to the study of isometric objects in n-dimensional Riemannian space with constant curvature. The formulae for the transformation of isometric figures, on which this study can be based, remain unchanged.

Contents

CHAPTER I

Elliptic Space

In order to make the main part of the present work more comprehensible, we consider it advisable to recall certain facts about the geometry of elliptic space. The present section therefore deals with them.

1. FOUR-DIMENSIONAL VECTOR SPACE

By a four-dimensional vector we mean any set of four real numbers (x_0, x_1, x_2, x_3). The numbers x_i are called the coordinates. The operations of addition, subtraction and multiplication by a real number are determined for these vectors in the normal way. That is, the sum (difference) of the vectors (x_i) and (y_i) is the vector $(x_i \pm y_i)$, while the product of the vector (x_i) and the number λ is the vector (λx_i).

Four dimensional vectors can be imagined as directed segments in four dimensional Euclidean space. Here the formally-introduced operations of addition, subtraction of vectors and multiplication by a number correspond to known geometric calculations for three-dimensional factors.

Apart from these operations, we can introduce the scalar products of two vectors, the vector product of three vectors and the mixed product of four vectors.

Indeed, the scalar product of the two vectors x and y will be the number

$$(xy) = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where x_i is the i-th coordinate of the vector x, and y_i is the i-th coordinate of y.

The vector product of the three vectors, x, y and z, taken in this order, will be designated the vector (xyz), the i-th coordinate of which is equal to the product of $(-1)^i$ with the determinant of the three by three minor of the matrix,

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{pmatrix}$$

which is obtained by eliminating the i-th column (the number of the column is determined by the subscript on the coordinates).

Finally, the mixed product of the four vectors, a, b, c and d is the number, equal to the determinant, composed of the vector's coordinates.

$$(abcd) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix}.$$

The vector* product clearly is not changed by cyclic

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* In the original it was erroneously claimed that the mixed product also has this property – Ed.

permutation of the multipliers. The vector and the mixed products are multiplied by (-1) when permutating two multipliers. It follows from this that the vector and mixed products are equal to zero when the multipliers are linearly dependent.

The scalar, vector and mixed products of the vectors are distributive with respect to each of the multipliers.

Let us note the following identities:

$$(abc)\, d = (abcd);$$

$$(abc)\,(a'b'c') = \begin{vmatrix} (aa') & (ab') & (ac') \\ (ba') & (bb') & (bc') \\ (ca') & (cb') & (cc') \end{vmatrix};$$

$$(abcd)\,(a'b'c'd') = \begin{vmatrix} (aa') & (ab') & (ac') & (ad') \\ (ba') & (bb') & (bc') & (bd') \\ (ca') & (cb') & (cc') & (cd') \\ (da') & (db') & (dc') & (dd') \end{vmatrix}.$$

The first follows directly from the definition of scalar and vector multiplication, and the third from the theorem of multiplication of determinants. To prove the second identity we should point out that both sides of it are linear with respect to each of the vectors, a, b ... c'. Hence it is enough to verify it for the coordinate vectors (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1). Here, if both sets of three vectors a, b, and c and a', b' and c' are identical except possibly for the order of the vectors, both sides of the identity are equal either to + 1 or to -1, and if the three sets are different, both sides are equal to zero. (The vector product of three different coordinate vectors is equal to the fourth coordinate vector with the appropriate sign).

The absolute value of the vector x is the length of the segment corresponding to it, and is calculated by the usual formula, just as for three dimensional vectors:

$$|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} = \sqrt{(x, \ x)}.$$

We will take a rotation of the vector space to mean a linear transformation

$$x_i' = \sum_j \alpha_i^j x_j, \quad i, \ j = 0, \ 1, \ 2, \ 3, \tag{*}$$

which does not change the absolute values of the vectors, i.e., for any x

$$|x| = |x'|.$$

Clearly, the scalar product of two vectors is invariant with respect to the rotations, since

$$(xy) = \frac{1}{4}\{(x+y)^2 - (x-y)^2\} = \frac{1}{4}\{|x+y|^2 - |x-y|^2\}.$$

During rotation (*) the coordinate vectors e_0 (1, 0, 0, 0), ..., e_3 (0, 0, 0, 1) change, respectively, to the vectors $(\alpha_0^0, \ \alpha_1^0, \ \alpha_2^0, \ \alpha_3^0), \ldots, (\alpha_0^3, \ \alpha_1^3, \ \alpha_2^3, \ \alpha_3^3)$. Taking the invariance of the scalar product into account, we therefore conclude that the matrix is orthogonal (α_i^j):

$$\alpha_0^i \alpha_0^j + \alpha_1^i \alpha_1^j + \alpha_2^i \alpha_2^j + \alpha_3^i \alpha_3^j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \tag{**}.$$

The determinant of the transformation (*) is

$$\Delta = \begin{vmatrix} \alpha_0^0 & \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \alpha_3^1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_0^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 \end{vmatrix} = \pm 1.$$

To see that this is so, we need only square Δ and make use of the orthogonality relationships (**). Let us agree to call the rotation proper when $\Delta = +1$ and improper when $\Delta = -1$. Let A be any rotation. Then

$$(Ax, \ Ay, \ Az) = \pm A(xyz);$$
$$(Aa, \ Ab, \ Ac, \ Ad) = \pm (abcd),$$

where the plus sign holds for proper rotations and the minus sign for improper ones. The second of these formulae is a simple corollary of the theorem of multiplication of determinants, since

$$(Aa, \ Ab, \ Ac, \ Ad) = (abcd)\,\Delta.$$

The first formula follows from the second, since for any vector u, on the one hand,

$$(Ax, \ Ay, \ Az, \ Au) = (Ax, \ Ay, \ Az)\,Au$$

and, on the other,

$$(Ax, \ Ay, \ Az, \ Au) = \Delta(xyzu) = \Delta(xyz)\,u = \Delta A(xyz)\,Au.$$

2. THE CONCEPT OF ELLIPTIC SPACE

Elliptic space can be defined as a complete three-dimensional Riemannian manifold with a constant curvature, homeomorphic to projective space. A

sphere in a four-dimensional Euclidean space is also a three-dimensional Riemannian manifold with constant curvature, and is therefore locally isometric to elliptic space. If, starting out from a certain point, the sphere is locally isometrically mapped on elliptic space of corresponding curvature, the entire sphere will cover it twice, with each pair of diametrically opposite points in the sphere corresponding to the same point in elliptic space. This makes it possible to imagine elliptic space in the form of a three-dimensional sphere, in which diametrically opposite points are identified. From now on we will use this illustrative model of elliptic space. We will consider the curvature of the space equal to unity, and the corresponding sphere will therefore have a unit radius.

Since a motion of elliptic space is an isometric transformation, and every isometric transformation of a sphere into itself is a rotation (proper or improper), a motion of the elliptic space on the spherical model should be regarded as a rotation of the sphere about its center. We should further point out that the lines of elliptic space, like the geodesics in the spherical model, are represented by great circles, while the planes are given by intersections of hyperplanes passing through the center of the sphere with the sphere itself.

Let us introduce coordinates in the elliptic space by assigning to each point the Cartesian coordinates for the corresponding point in the unit sphere. Such coordinates in elliptic space are called Weierstrass coordinates. On account of the double-valued nature of the isometric representation of elliptic space on the sphere, they can only be determined up to the

sign. If, instead of the whole space, we consider part of it, having removed a plane, for instance, x = 0, as we will often do, this non–single–valued nature can be eliminated by the further requirement $x_0 > 0$.

Each plane in elliptic space in Weierstrass coordinates is given by the linear homogeneous equation

$$a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0,$$

where a_i are constants, which are not all equal to zero. Each straight line is given by an independent system of two such equations:

$$a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0;$$
$$b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 = 0.$$

These equations clearly enable us to define a plane by means of the scalar products

$$(ax) = 0 \qquad\qquad\qquad (*)$$

and, correspondingly,

$$(ax) = 0, \ (bx) = 0. \qquad\qquad (**)$$

This way of specifying a straight line makes it possible to define it parametrically:

$$x = \rho\,(c + dt),$$

where c and d are vector constants, t is a parameter and ρ is a normalizing factor defined by the condition $x^2 = 1$. As c and d we can take any two different solutions of equations (**). Similarly, the plane (*) can

be given by the equation in parametric form

$$x = \rho\,(c + ud + ve),$$

where c, d and e are three independent solutions of the equation (*), u, v are parameters and ρ is a normalizing factor defined by the same condition.

Let us express the line element of elliptic space in Weierstrass coordinates. Since these coordinates are equal (up to the sign) to the Cartesian coordinates of an isometrically–corresponding point on the sphere, the line element of elliptic space in Weierstrass coordinates coincides with the line element of the sphere in Cartesian coordinates, i.e.,

$$ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

In concluding this section we consider the projective model of elliptic space which we are also going to use. To do this, we add ideal points to the Euclidean space $x_0 = 1$. The projective space obtained will be designated P_0. Let us now compare each point in elliptic space with Weierstrass coordinates x_i with a point in projective P_0 with homogeneous coordinates $\bar{x}_i$ by the formula

$$\bar{x} = \frac{x}{(xe_0)},$$

where e_0 is the unit coordinate vector (1, 0, 0, 0). This mapping is clearly one-to-one. It can be illustrated by a projection of a unit sphere from its center to the hyperplane $x_0 = 1$.

This mapping of elliptic space on projective space is noteworthy in many respects. It is geodesic in the sense that straight lines of elliptic space become straight lines in projective space. The motions of elliptic space correspond to projective transformations which leave an imaginary quartic surface $\bar{x}^2 = 0$ invariant. This fact stems from the invariance of the scalar product $(\mathbf{x}, \mathbf{x})$ with respect to rotations.

This enables us to imagine elliptic space as projective space in which the role of motions is played by projective transformations, which leave the imaginary quartic surface $\bar{x}^2 = 0$ fixed.

We should further point out that if the plane $x_0 = 0$ is removed from elliptic space, the remaining part of the space is mapped into the Euclidean space $x_0 = 1$ by this method in a one-to-one manner. This Euclidean space is osculating with respect to the elliptic space at the point $(1, 0, 0, 0)$. Indeed, the line element of the Euclidean space is $d\bar{s}^2 = d\bar{x}^2$. Since at point $(1, 0, 0, 0)$ $(xe_0) = x_0$ is equal to unity and stationary,

$$ds^2 - ds^2 = 0, \ \delta\,(d\bar{s}^2 - ds^2) = 0.$$

This means that Euclidean space at this point is osculating.

3. CURVES IN ELLIPTIC SPACE

The definition of the length of a curved line in elliptic space is the same as in Euclidean space. It is the limit of the lengths of open polygons inscribed into a curve when the links become arbitrarily small. The length of a link of an open polygon with ends at

the points (t) and $(t + \Delta t)$ on the curve $x = x(t)$ is

$$\simeq |x(t + \Delta t) - x(t)| \qquad (*)$$

In the same way as for Euclidean space curves as represented by ordinary vectors, we derive from this the formula

$$s = \int \sqrt{x'^2} dt.$$

We will call the parametrization of the curve natural if arc length serves as the parameter.

Let γ be a curve in elliptic space, and let

$$x = x(s)$$

be its natural parametrization. Let us ascertain the properties of the vector $\tau = x'(s)$.

First, τ is a unit vector, since

$$s = \int \sqrt{\tau^2} ds$$

and, consequently, $\tau^2 = 1$.

Now let us consider the straight line given by the equation in parametric form

$$\rho x = x(s) + t\tau(s)$$

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* We will use the sign $\simeq$ to mean an equality with accuracy up to the magnitude of the lowest degree of smallness, not appearing in the given case Δt^2.

(parameter t). It is a tangent to the curve γ at the point (s). Indeed, it passes through the point (s) $(t = 0)$. Let us calculate the distance between the point (s $+ \Delta s$) on the curve and the straight line. It is not greater than the distance between the given point on the curve and the point $x(s) + \Delta s \tau$ (s) on the straight line. But the distance between these points

$$\simeq \mid x\,(s + \Delta s) - x\,(s) - \tau\,(s)\,\Delta s \mid$$

is of the order of Δs^2. It follows from this that the straight line is a tangent. In view of this we will call the vector τ a unit tangent vector of the curve.

Let the curves γ' and γ'' originate from a common point P, and let

$$x = x'(s), \quad x = x''(\sigma)$$

be their natural parametrizations. Let us find the angle between the curves at the point P. By definition, the angle between the curves is the angle between the corresponding curves in osculating Euclidean space at the point P.

Let $P \equiv (1, 0, 0, 0)$. The curves corresponding to γ' and γ'' in osculating Euclidean space at point P are then given by the equations

$$y = \frac{x'}{(x'e_0)}\,, \quad y = \frac{x''}{(x''e_0)}$$

(see para. 2). Because of the stationary nature of $(x'e_0)$ and $(x''e_0)$ at point P, the unit tangent vectors of these curves will be τ' and τ'', and, therefore,

$$\cos \vartheta = (\tau' \tau'').$$

Since scalar multiplication of the vectors is invariant with respect to the motions of space, this formula is valid, no matter where P is.

Let us calculate the curvature k of the curve γ: $x = x(s)$ at an arbitrary point. By definition, the curvature of γ at the point P is the curvature of the corresponding curve in osculating Euclidean space. Let $P \equiv$ (1, 0, 0, 0). The corresponding curve in osculating Euclidean space is given by the equation

$$y = \frac{x(s)}{(\dot{x}(s)\,e_0)}.$$

As is well known, its curvature is expressed by the formula

$$k^2 = \frac{y'^2 y''^2 - (y' y'')^2}{(y'^2)^3}.$$

If the expression for y in terms of x is substituted into this formula and the following relationships are used:

$$x^2 = 1, \quad xx' = 0, \quad x'^2 = 1, \quad xx'' = -1, \quad x = e_0,$$

we obtain

$$k^2 = x''^2 - 1.$$

On account of the invariance of this formula with respect to motions, it is not only valid at the point (1, 0, 0, 0), but at any point.

Let us find the osculating plane of the curve γ at the point (s). To do this we will consider the plane given by the equation in parametric form:

$$\rho x = x(s) + ux'(s) + vx''(s)$$

(parameters u, v).

Obviously it passes through the point (s) on the curve (u $= v =$ 0). The distance between the point (s $+ \Delta s$) on the curve and this plane is no greater than the distance to the point

$$\frac{1}{\rho}\left(x(s) + \Delta sx'(s) + \frac{\Delta s^2}{2}x''(s)\right).$$

And this distance

$$\simeq \left| x(s + \Delta s) - x(s) - \Delta sx'(s) - \frac{\Delta s^2}{2}x''(s)\right|$$

is of the order of at least Δs^3. It follows from this that the plane is osculating.

The straight line $\rho x = x(s) + \lambda x''(s)$ (parameter λ) is the principal normal of the curve. Indeed, it lies in the osculating plane. Further, $\rho^2 = 1 - 2\lambda + 0\,(\lambda^2)$, and, consequently, at $\lambda = 0$

$$x_\lambda' = x''(s) + x,$$

from which it follows that $x_\lambda' x' = 0$, i.e., the vector x_λ' is perpendicular to the tangent vector of the curve, and the straight line is the principal normal.

The unit vector ν , defined by the condition

$$x'' + x = \lambda \nu,$$

is called the unit vector of the principal normal of the curve. The factor λ is simple in meaning — it is the curvature of the curve. Indeed, by squaring both sides of the equality we obtain

$$\lambda^2 = x''^2 - 1 = k^2.$$

We should point out in conclusion that x(s + Δ s) can be conveniently represented with an accuracy up to magnitude of the order of Δs^2 as

$$x(s + \Delta s) = x + \Delta s \tau + \frac{\Delta s^2}{2}(k\nu - x).$$

The vectors x, τ, ν are mutually perpendicular.

4. SURFACES IN ELLIPTIC SPACE

Let us assume we have the surface F in elliptic space given by the equation

$$x = x(u, v).$$

A tangent vector at the point P of this surface will mean a tangent vector of any curve on the surface originating from P. In particular, x_u and x_v are tangent vectors.

The plane given by the equation in parametric form

$$ox = x + \alpha x_u + \beta x_v \quad \text{(parameters} \quad \alpha, \beta),$$

is the tangent plane of the surface, which is easily proved by the argument given for the tangent curve in para. 3.

It follows easily from the properties of vector and mixed products of vectors that the vector

$$n = (x\ x_u\ x_v)$$

is perpendicular to all tangent vectors $(\lambda x_u + \mu x_v)$ of the surface at the point (u, v). In view of this we will term the vector n the surface normal vector. The unit surface normal vector will be designated ξ.

Two fundamental forms are introduced for surfaces in elliptic space in the same way as for Euclidean space, namely, the first fundamental form is the line element of the surface

$$ds^2 = dx^2 = Edu^2 + 2Fdudv + Gdv^2,$$

and the second fundamental form is

$$- (dxd\xi) = (d^2x,\ \xi) = Ldu^2 + 2Mdudv + Ndv^2.$$

The lengths of the curves, the angles between them, the geodesic curvature of the curve and the Gaussian (intrinsic) curvature of the surface are expressed by means of the first fundamental form in the normal way, as for Euclidean surfaces.

Let us consider the second fundamental form of the surface in more detail.

Let us assume we have a curve on the surface

$$u = u\ (s),\ v = v\ (s),$$

where s is an arc along the curve. Let us find its curvature. From the formula in para. 3

$$x_{ss}'' = k\nu - x,$$

where k is the curvature of the curve and ν is the unit vector of its principal normal. By multiplying this equality by ξ and noting that the vectors x and ξ are perpendicular, we obtain

$$(x_{ss}''\xi) = k \cos \vartheta,$$

where ϑ is the angle between the principal normal of the curve and the normal of the surface.

On the other hand, since the vectors x_u and x_v are perpendicular to ξ,

$$(x_{ss}''\xi) = (x_{uu}\xi) u'^2 + 2 (x_{uv}\xi) u'v' + (x_{vv}\xi) v'^2 =$$
$$= Lu'^2 + 2Mu'v' + Nv'^2.$$

As a result we obtain the following formula:

$$k \cos \vartheta = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}. \qquad (*)$$

The right-hand side of this formula has simple geometrical meaning. It is the curvature of the normal section of the surface $(\vartheta = 0)$.

By defining the principal curvatures of the surface as the extreme values of normal curvatures at a given point, we can find an equation for them by the well-known method

$$\begin{vmatrix} L - kE, & M - kF \\ M - kF, & N - kG \end{vmatrix} = 0,$$

from which we obtain the following expression for

the product of the principal curvatures:

$$k_1 k_2 = \frac{LN - M^2}{EG - F^2} \cdot$$

The product of the principal curvatures is called the extrinsic curvature of the surface. In elliptic space the extrinsic curvature of the surface K_e differs from the intrinsic curvature (para. 5.), which is calculated in the known way from the coefficients of the first fundamental form and their derivatives.

Since we have formula (*) for the normal curvature of the surface, it would be possible to introduce asymptotic lines and lines of curvature, as is done for Euclidean space surfaces, and to prove the corresponding theorems, but we will not do this, since these theorems are hardly used at all in the subsequent argument.

We will show in conclusion that the principal extrinsic characteristic of a geodesic on surfaces in Euclidean space — the fact that the principal normal coincides with the surface normal — is true of geodesics on the surfaces in elliptic space.

Let γ be a geodesic on the surface F of elliptic space and P be a point on it. Without loss of generality, we can suppose that $P \equiv (1, 0, 0, 0)$. Let us plot the osculating Euclidean space at the point P, as in para. 1. Let F be the surface and $\bar{\gamma}$ be a curve on it, corresponding to F and γ. Clearly, $\bar{\gamma}$ at P on F has a geodesic curvature equal to zero. Hence the principal normal of the curve $\bar{\gamma}$ at P coincides with the normal of the surface

$$\overline{(\overline{v}\overline{x_u})} = 0, \quad \overline{(\overline{v}\overline{x_v})} = 0.$$

Substituting into this

$$\overline{x} = \frac{x}{(xe_0)}, \quad \overline{v} = \frac{1}{k}\,\overline{x}''$$

and noting that $x = e_0$ at P, we obtain

$$(vx_u) = 0, \quad (vx_v) = 0,$$

where

$$v = (x'' + x)\frac{1}{k}$$

is the unit vector of the principal normal of γ at P. Since, moreover, $xv = 0$, the vectors v and ξ either coincide or are oppositely directed. Hence the statement is proved.

5. FUNDAMENTAL EQUATIONS IN THE THEORY OF SURFACES IN ELLIPTIC SPACE

Let F be a surface in elliptic space. At the point (x) on the surface the vectors, x, x_u, x_v, and ξ form a basis, since $(xx_ux_v\xi) \neq 0$. Hence any vector at this point can be expressed linearly in terms of x, ..., ξ. In particular,

$$x_{uu} = A^1_{11}x_u + A^2_{11}x_v + C_{11}x + \lambda_{11}\xi;$$
$$x_{uv} = A^1_{12}x_u + A^2_{12}x_v + C_{12}x + \lambda_{12}\xi;$$
$$x_{vv} = A^1_{22}x_u + A^2_{22}x_v + C_{22}x + \lambda_{22}\xi;$$
$$\xi_u = B^1_1x_u + B^2_1x_v + D_1x + H_1\xi;$$
$$\xi_v = B^1_2x_u + B^2_2x_v + D_2x + H_2\xi.$$

The coefficients of these formulae can be expressed in terms of the coefficients of the first and second

fundamental forms of the surface. In fact, by multiplying these equations scalarly by ξ, we obtain

$$\lambda_{11} = L, \quad \lambda_{12} = M, \quad \lambda_{22} = N, \quad H_1 = H_2 = 0.$$

By multiplying the equations by x and observing that

$$x^2 = 1, \quad x_{uu}x = -E, \quad x_{uv}x = -F, \quad x_{vv}x = -G, \quad x_{\cdot u}^{\cdot} = x_{\cdot v}^{\cdot} = 0,$$

we obtain

$$C_{11} = -E, \quad C_{12} = -F, \quad C_{22} = -G, \quad D_1 = D_2 = 0.$$

To obtain the coefficients A_{11}^1 and A_{11}^2 we multiply the first equation by x_u and x_v. Observing that $x_{uu}x_u = \frac{1}{2}E_u$ $x_{uu}x_v = F_u - \frac{1}{2}E_v$, we obtain

$$\frac{1}{2}E_u = EA_{11}^1 + FA_{11}^2,$$

$$F_u - \frac{1}{2}E_v = FA_{11}^1 + GA_{11}^2.$$

The Christoffel symbols of the second kind, Γ_{11}^1 and A_{11}^2 are the unique solution of this system. Hence,

$$A_{11}^1 = \Gamma_{11}^1, \quad A_{11}^2 = \Gamma_{11}^2.$$

Similarly we conclude that

$$A_{12}^1 = \Gamma_{12}^1, \quad A_{12}^2 = \Gamma_{12}^2;$$

$$A_{22}^1 = \Gamma_{22}^1, \quad A_{22}^2 = \Gamma_{22}^2.$$

To determine the coefficients B_2^1 and B_2^2 we multiply

the last equality by x_u and x_v. Then we obtain:

$$-M = B_2^1 E + B_2^2 F,$$
$$-N = B_2^1 F + B_2^2 G.$$

And this is just the system from which we find the coefficients of the corresponding derivation formulae for Euclidean surfaces. We can draw a similar conclusion with regard to the coefficients B_1^1 and B_1^2.

Thus, for surfaces in elliptic space there is the following system of derivation formulae:

$$x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v - Ex + L\xi;$$
$$x_{uv} = \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v - Fx + M\xi;$$
$$x_{vv} = \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v - Gx + N\xi;$$
$$\xi_u = B_1^1 x_u + B_1^2 x_v;$$
$$\xi_v = B_2^1 x_u + B_2^2 x_v,$$

where the coefficients Γ and B are expressed in terms of E, F, G, L, M and N in exactly the same way as the corresponding coefficients of the derivation formulae for Euclidean space surfaces.

The conditions which the coefficients of the first and second fundamental forms have to satisfy may be obtained in the same way as for Euclidean space surfaces. Specifically, if in the identities

$$(\xi_u)_v - (\xi_v)_u = 0,$$
$$(x_{uu})_v - (x_{uv})_u = 0,$$
$$(x_{vv})_u - (x_{uv})_v = 0$$

the expressions in parentheses are replaced in accordance with the derivation formulae, and after formal differentiation this substitution is used again, we obtain

$$\alpha_{11}x_u + \alpha_{12}x_v + \alpha_{10}x + \alpha_1\xi = 0,$$
$$\alpha_{21}x_u + \alpha_{22}x_v + \alpha_{20}x + \alpha_2\xi = 0,$$
$$\alpha_{31}x_u + \alpha_{32}x_v + \alpha_{30}x + \alpha_3\xi = 0.$$

Since x_u, x_v, x and ξ are independent vectors,

$$\alpha_{ij} = 0, \quad \alpha_i = 0.$$

We should point out that even if there had not been any terms containing x in the first three derivation formulae, we would have obtained the same expressions for α_{11} and α_{12} . But since in the case of Euclidean space $\alpha_{11} = 0$ and $\alpha_{12} = 0$ are Peterson-Codazzi equations, the coefficients of the first and second fundamental forms of surface in elliptic space must also satisfy the Peterson-Codazzi equations.

Without more detailed analysis we will point out that there is only one new relation among the remaining ten equations. It links the intrinsic (Gaussian) and extrinsic curvature of the surface:

$$\frac{LN - M^2}{EG - F^2} = K_i - 1.$$

In conclusion we should point out that these three relationships between coefficients of the fundamental forms

$$E \, du^2 + 2F \, du \, dv + G \, dv^2,$$
$$L \, du^2 + 2M \, du \, dv + N \, dv^2,$$

are sufficient to determine a surface locally for which these forms are the first and second fundamental forms, respectively. This surface is unique up to motions in space.

CHAPTER II

Convex Bodies and Convex Surfaces in Elliptic Space

The main part of this book deals with study of iso-
metric general convex surfaces in elliptic and Euclid-
ean space. In view of this, we will now consider some
of the properties of convex bodies and convex sur-
faces in elliptic space used later on in the book.

1. THE CONCEPT OF A CONVEX BODY

As was shown in Chapter I, a four-dimensional unit
sphere admits a one-to-two locally-isometric map-
ping onto elliptic space with unit curvature. This
mapping is unique up to motions.

We will say the body in elliptic space is convex if
it is an image of a convex body on the unit sphere,
under this mapping. Clearly, this definition is invar-
iant with respect to motions in elliptic space and does
not depend on a specific representation of the sphere.

A convex body on a sphere in four-dimensional
space can be defined as the intersection of a convex
cone with the apex at the center of the sphere with
the surface of the sphere. Since a convex cone has a
plane of support at the apex, a convex body on the
sphere is entirely contained in a hemisphere deter-
mined by the plane of support. If the apex of the cone

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is a strictly conical point in the sense that one of its planes of support has no other points in common with the cone, then the convex body defined by this cone lies inside the open hemisphere determined by the plane.

Now let us turn to the projective model of elliptic space. If we take the cone's plane of support as the plane $x_0 = 0$, and consider the cone to be inside the half-space $x_0 > 0$, then on $x_0 = 1$ a convex body in the elliptic space looks like a Euclidean-convex body, possibly including ideal points. In order to gain a complete idea of it we should point out the following.

One of the four possibilities occurs at the apex of the convex four-dimensional cone, a) there is a plane of support without any other points in common with the cone except for its apex, and in this case we call the apex a strictly conical point; b) there is a plane of support which has a straight line in common with the cone, but has no other points in common; c) there is a plane of support which has a two-dimensional plane in common with the cone, but no other points; d) the entire plane of support belongs to the cone. In case (a) on the projective model $x_0 = 1$, the convex body looks like a bounded convex body; in the case (b) it is a Euclidean convex cylinder with a bounded cross-section supplemented by a point at infinity; in case (c) the convex body consists of a layer between two parallel planes, supplemented by a straight line at infinity intersected by the planes; finally, in case (d) the convex body represents the entire space.

In cases (b), (c) and (d) we are dealing with convex bodies which are degenerate in the accepted sense; their surfaces are simple and study of them is not of

interest at this point. In view of this we shall only go on to consider convex bodies and their surfaces which are obtained in case (a), i.e., when the apex of the projected cone is strictly conical.

Thus, the convex bodies in elliptic space under consideration in the spherical model are convex bodies lying strictly inside one hemisphere, and in the projected model, they are bounded by Euclidean convex bodies.

When considering convex bodies in elliptic space, it is more convenient to single out a plane which does not intersect with the body. On the spherical model this plane could be the intersection of the plane of support of the projecting cone and the sphere. A plane at infinity is correspondingly singled out on the projected model. Here, on the spherical model we consider convex bodies lying in an open hemisphere $x^2 = 1$, $x_0 > 0$, while on the projected model we consider bounded convex bodies of Euclidean space $x_0 = 1$.

When the plane $x_0 = 0$ has been removed from elliptic space, the straight lines and planes which are left have the properties of Euclidean straight lines and planes. This makes it possible to introduce various concepts for convex bodies in elliptic space in the same way as for convex bodies in Euclidean space, in particular the concept of the plane of support and the tangent cone, and to prove the corresponding theorems.

2. CONVEX SURFACES IN ELLIPTIC SPACE

Just as in Euclidean space, we will take a convex

surface in elliptic space to mean an area on the boundary of a convex body. The entire boundary of the convex body is called a complete convex surface.

The distance between two points on the convex surface is taken to mean the greatest lower bound of the length of curves on the surface joining these points. A curve, whose length is equal to the distance between its end points on the surface is called a segment. (A curve in elliptic or in Euclidean space of length equal to the distance between its endpoints in the metric of the space will be called a line segment—Ed.)

Any two points on a complete convex surface in elliptic space can be joined by a segment. On an incomplete convex surface there may be points which cannot be joined by a segment, but every point has a neighborhood, any two points of which may be joined by a segment on the surface. Convex surfaces in Euclidean space conform to Buseman's theorem, which states that if a curve connects two given points on a complete convex surface, lies outside the body bounded by it and does not lie entirely on the surface, then it has a longer length than the distance between these points on the surface. In elliptic space, Buseman's theorem is, generally speaking, invalid in the form in which we have just stated it. Examples are not hard to find. But if the points on the surface are sufficiently close together the theorem remains valid.

To make this clear, let us turn to the projective model of elliptic space. Here the convex body is represented by a bounded Euclidean convex body, and its surface is represented by a closed convex surface. Let us use δ to designate the greatest lower bound of the lengths of curves in the metric of elliptic space,

joining the surface and the plane at infinity. Now let the distance s between points A and B on the surface be less than δ . We will demonstrate that any curve joining point A and B outside the body bounded by the surface, and not lying entirely on it, is of greater length than s. Let us assume that this is untrue, and consequently, that there are curves which are of length not greater than s.

Let us consider all the curves of length not greater than s joining points A and B, which do not lie entirely on the surface. They clearly do not intersect the plane at infinity. Among these curves one is at least as short as all of the others. It can be supposed that it does not lie entirely on the surface. Indeed, if it is less than s in length, it cannot lie on the surface, since the intrinsic distance between its ends is equal to s. If it is the same length as s, then, as assumed, it will be a curve of s length joining the points A and B and will not lie entirely on the surface. On the other hand, each component of the part of this shortest curve which does not lie on the surface is a line segment with ends on the surface. Since such a component cannot be inside the body, it must lie on the surface. This contradiction proves the statement.

Buseman's theorem allows us to extend Liberman's lemma on geodesics to cover convex surfaces in the following form.

Let γ be a geodesic on a convex surface (i.e. a curve such that each sufficiently small part of it is a segment). Let us take the point O inside the convex body bounded by the surface. Let us join the point O to all points on the geodesic by line segments inside the body bounded by the surface. Then, if the conical

surface formed by these segments is developed onto an elliptic plane, γ will correspond to a convex curve $\bar\gamma$, with its convexity turned away from the area covered by the cone in the process. Indeed, it this were not the case, there would be arbitrarily many points near each other on $\bar\gamma$ which could be joined by a small line segment δ outside the area covered by the development of the cone. If the curve γ_δ, corresponding to the segment γ, is plotted on the cone projecting the geodesic γ , it will be shorter than the segment of the geodesic corresponding to it, which contradicts Buseman's theorem.

This lemma has numerous corollaries. Without enumerating them, we shall merely point out that they show the existence and continuity from the right of a right semi-tangent at each point of a geodesic. A similar statement holds for left semi-tangents.

Let S be a point on the convex surface in elliptic space and let γ be a geodesic extending from S. Let us take the point X close to S on γ and join it to S by a line segment. Let s be the distance between the points S and X along the geodesic, δ the spatial distance between these points, and ϑ the angle which is formed by the semi-tangent to γ at S with the line segment SX. Then, when X→S, the ratio $s/\delta \to 1$, and $\vartheta \to 0$. This property of geodesics is a simple corollary of the existence and one-sided continuity of the semi-tangent to the geodesic.

In the geodesic mapping of elliptic space on Euclidean space (Chapter I) the tangent cones of convex surfaces correspond to each other. Since here the property of having a semi-tangent is retained by corresponding curves and the semi-tangent curves

originating from the given point are generators of the tangent cone on convex surface of Euclidean space, this property is also possessed by convex surfaces in elliptic space. In particular, semi-tangent geodesics originating from the given point on a convex surface in elliptic space are generators of the tangent cone.

Let us now consider the total extrinsic curvature of convex surfaces in elliptic space. In Euclidean space the total extrinsic curvature of the set M on the convex surface is defined as the area of the spherical image of this set. In elliptic space this definition of curvature meets with difficulties on account of the absence of a parallel translation to transfer the normals to one point.

This difficulty is overcome in Aleksandrov's work [1] in the following way. The set M is broken up into a finite number of subsets m_k. The point P_k is taken in each m_k and at this point we plot an osculating Euclidean space corresponding geodesically to constant-curvature space (as in Chapter I). Let $\omega\,(m_\kappa)$ be the extrinsic curvature of the set corresponding to m_k on the convex surface in Euclidean space. One proves that if the diameters of the set m_k decrease infinitely, $\sum_\kappa \omega\,(m_\kappa)$ tends to a definite limit, which does not depend upon the way M is broken down into m_k. This limit is, by definition, the extrinsic curvature of the surface at the set M.

Regarding the extrinsic curvature of convex surfaces in Euclidean space determined in this way, Aleksandrov in the work quoted establishes a number of properties, in particular, the complete additivity of the ring of Borel sets.

In a case where the convex surface is regular, the above-given definition of extrinsic curvature results in the following expression:

$$\omega\,(M) = \int\limits_{M} k_1 k_2 \, ds,$$

where k_1 and k_2 are the principal curvatures, and the integration is made with respect to the surface area.

3. THE DEVIATION OF A SEGMENT ON A CONVEX SURFACE FROM ITS SEMI-TANGENT AT ITS INITIAL POINT

Let F be a convex surface in elliptic space, O be a point on this surface and X a point on the surface close to O. Let us join the points X and O by the segment γ and use t to denote the semi-tangent to it at the point O. Let us mark off a line segment OY equal to s, the length of the segment γ, on the semi-tangent t. We wish to calculate the length δ of the segment XY and to establish its direction.

In the case of convex surfaces in Euclidean space, the following are known [2, p. 24]:

1. The direction of the segment XY as a point on the unit sphere belongs to the convex hull of the spherical image of the segment γ.

2. $\delta/s \to 0$ when X $\to$ 0.

3. The angle, ϑ, between the line segments OY and YX approaches $\frac{\pi}{2}$ as x $\to$ 0.

We will now establish similar properties for con-
vex surfaces in elliptic space.

Let us take Euclidean space osculating at the point
O with respect to the elliptic space in question and
geodesically corresponding to it. Let us transfer the
given figures to this space, keeping the same nota-
tion, but with a line above the symbols.

The proof of the first statement for surfaces in
Euclidean space is given, first, for polyhedra, and in
this case is as follows: the faces along which the geo-
desic $\bar{\gamma}$ passes, are turned into the plane of the first
link by turning them about the edges joining them.
Thus, the open geodesic polygon $\bar{\gamma}$ is transformed to
a line segment of length s, which has the direction of
the first link, while the end of the open polygon (the
point X) moves along a smooth curve $\tilde{\gamma}$, comprised
of the arcs of great circles, and every point as $\tilde{\gamma}$ lies
in a supporting plane of the surface along $\bar{\gamma}$. Thus,
all of the tangents to $\tilde{\gamma}$ are external normals of the
surface along $\bar{\gamma}$, whence property 1 follows.

A similar transformation of a geodesic open poly-
gon on a polyhedron in elliptic space is clearly easy
to accomplish. The trajectory $\tilde{\gamma}$ of the point X cor-
responds to a smooth curve in the osculating Euclid-
ean space, which intersects the planes of support of
the polyhedron $\overline{F}$ along $\bar{\gamma}$ at angles close to right
angles, and this deviation from a right angle may be
made as small as is required, by making s small.
It follows from this that the direction of the segment

$\overline{XY}$ is located in a ε' neighborhood of the convex hull of the spherical image of $\overline{\gamma}$, with $\varepsilon' \to 0$, when $s \to 0$.

The extension from a polyhedron to an arbitrary convex surface is made by approximating the surface by a polyhedron, and is not essentially different from the corresponding change in Euclidean space. The final result can be formulated in the following way.

The direction of the segment $\overline{XY}$ of the image of $\overline{XY}$ in osculating Euclidean space at the point O, considered as a point on the unit sphere, belongs to an $\varepsilon(s)$-neighborhood of the convex hull of the spherical image of $\overline{\gamma}$ where $\overline{\gamma}$ is the image of γ, in the osculating Euclidean space at O. $\varepsilon(s) \to 0$, as $s \to 0$.

We will now demonstrate that the angle $\overline{\vartheta}$ between the line segments $\overline{OX}$ and $\overline{YS}$ tends to $\frac{\pi}{2}$, when $s \to 0$.

Let us assume that this is untrue, and that $X_\kappa \Big| \frac{\pi}{2} -$

$- \vartheta_\kappa | > a > 0$ for a certain sequence of points which converges to O. Without loss of generality, it can be supposed that the semi-tangents to γ_κ at O converge to a generator t_0 of the tangent cone at the point O. Consider a geodesic sector V with its apex at the point O containing the direction $\bar{t}_0$,* and so small that the spherical image of its image $\overline{V}$ on $\overline{F}$ is contained in a small neighborhood of the spherical image formed by the normals of the planes of support passing through $\bar{t}_0$.

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*That is, containing a quasi-geodesic having t_0 as a semi-tangent at O. —Ed.

The geodesics γ_k pass inside V, for large k, hence the directions of the segments $\overline{X_k Y_k}$ pass through every neighborhood of the spherical image of the sector $\overline{V}$. Consequently, the angle between $O\overline{X_k}$ and $\overline{X_k Y_k}$ comes arbitrarily near to a right angle, despite what was assumed.

It follows from the properties of osculating Euclidean space that the angle ϑ between the sectors OX and XY also tends to $\frac{\pi}{2}$, when $s \to 0$.

Finally, we will show that as $s \to 0$, $\delta/s \to 0$. Let us assume that this is not true. There is then a sequence of points X_k which is such that $s_k \to 0$ and $\delta_\kappa/s_\kappa > m > 0$. Let us repeat for this sequence the previous plotting and take sector V with a small angle at the apex. Without loss of generality it can be supposed that the rays through OX_k converge to a generator t'_0 of the tangent cone. Since $\delta_\kappa/s_\kappa > m$, and the segments $X_k Y_k$ and $X_k O$, according to what has been proved are nearly perpendicular, the angle between OX and OY is always greater than a certain $\beta > 0$.

Hence, it follows from this that the generators t_0 and t'_0 also form an angle greater than β. This is impossible if the angle of sector V was taken small enough. We are faced with a contradiction. The statement is therefore proved.

4. MANIFOLDS OF CURVATURE NOT LESS THAN K. A.D. ALEKSANDROV'S THEOREM

Let R be a two-dimensional metric manifold, i.e., a two-dimensional manifold which is a metric space. The metric ρ of the manifold is said to be intrinsic when given any pair of points X and Y of R, the distance between them $\rho(X, Y)$ is equal to the greatest lower bound of the lengths of the curves joining these points.

Let X by a point in R and let γ_1, and γ_2 be curves originating from it. Let us take the arbitrary points X_1 and X_2 on them, respectively, and construct a triangle on an Euclidean plane with sides $\rho(X, X_1)$, $\rho(X, X_2)$,, and $\rho(X_1, X_2)$. Let $\alpha(X_1, X_2)$ be the angle of this triangle lying opposite the side $\rho(X_1, X_2)$. Then,

$$\lim_{X_1, X_2 \to X} \alpha(X_1, X_2).$$

is called an angle between the curves γ_1, and γ_2 at the point X. In this sense an angle exists between any two curves at their point of intersection.*

The manifold R with the intrinsic metric is called a manifold of curvature not less than K, provided for any geodesic triangle the sum of its angles is not less than for a triangle with the same sides on a plane in space of constant curvature K.

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*Apparently the author's meaning here is that any real number which is a limit point of (X_1, X_2) as X_1, $X_2 \to X$ is to be called an angle between γ_1 and γ_2 at X. —Ed.

Manifolds of curvature not less than K have been introduced by A.D. Aleksandrov [3], who formulated a theory of these manifolds. We will recall the fundamentals of this theory since they bear on what will be described later.

If a manifold of curvature not less than K is Riemannian, its Gaussian curvature is not less than K. Conversely, each such Riemannian manifold is a manifold of curvature not less than K.

All convex surfaces in elliptic space of curvature K are manifolds of curvature not less than K.

Let γ_1 and γ_2 be segments originating from the point X on the manifold of curvature not less than K. By definition, the angle between them at X is called lim $\alpha(X_1, X_2)$, when X_1 and $X_2 \to X$. It turns out that for segments the limit of $\alpha(X_1, X_2)$ simply* exists. In view of this, from now on we will take the angle between any curves γ_1 and γ_2 to be the (usual—Ed.) limit of the angle $\alpha(X_1, X_2)$ as X_1, $X_2 \to X$. In this sense, the angle between curves does not necessarily exist.

For the curves to form an angle in the sense of this definition, it is both necessary and sufficient for each of them to form an angle with itself, which will obviously be equal to zero. Such curves are said to have a definite direction. The existence of a definite direction at a given point for a curve on a convex surface in elliptic space is equivalent to the existence of a semi-tangent at this point.

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*That is, in the usual sense. -Ed.

For manifolds of curvature of not less than K we now introduce the concept of intrinsic curvature as an additive set function such that an open geodesic triangle takes the values

$$\alpha + \beta + \gamma - \pi,$$

where α, β, γ are the angles of the triangle. If the manifold of curvature not less than K is Riemannian, the intrinsic curvature defined in this way is simply the integral curvature, i.e.,

$$\iint K_i \, d\sigma,$$

where K_i is the Gaussian curvature of the manifold, and the integration is with respect to area.

The concept of area in the manifold of curvature not less than K is introduced in the following way. Let G be a sector in the manifold of curvature not less than K and let Δ be any system of mutually disjoint triangles in it. Let us construct for each triangle $\delta \subset \Delta$ a plane triangle with the same sides. Let $\sigma(\delta)$ be the area of this triangle. Then, by the area we mean

$$\lim \sup \sum_{\Delta} \sigma(\delta)$$

on condition that the sides of the triangles decrease unlimitedly. In the case of Riemannian manifolds, this definition provides the usual area.

The Gauss theorem on the relationship between intrinsic and extrinsic curvature can be extended to the convex surfaces in a space of constant curvature

K. Indeed, if G is any section on a convex surface, $\varphi(G)$ is its extrinsic curvature, $\omega(G)$ is the intrinsic curvature and the $\sigma(G)$ is the area, then

$$\omega(G) = \varphi(G) + K\sigma(G).$$

Let two curves γ and γ' with definite directions at the point X but without any other points in common originate from the point X on the manifold of curvature not less than K. These curves split a neighborhood of point X into two "sectors." Let V be one of them. Let us draw segments γ_k from X into V, and number them in order of sequence from γ to γ'. Let $\alpha_1, \ldots \alpha_n$ be the angles between consecutive segments. The magnitude

$$\theta(\gamma, \gamma') = \sup \sum_k \alpha_k,$$

where sup is taken over all the systems of these segments, is called the vector angle between γ and γ' (angle with the side V). The sector angle has the usual properties. That is, if the curves γ_1, γ_2 and γ_3, originating from the point X determine three vectors V_{12}, V_{23} and V_{13}, and $V_{12} + V_{23} = V_{13}$, then $\theta(\gamma_1, \gamma_2) + \theta(\gamma_2, \gamma_3) = \theta(\gamma_1, \gamma_3)$. In the case of a convex surface in space of curvature K, the sector angle coincides with the angle between the semitangents γ and γ' on the development of the tangent cone at point X.

Let γ be a simple curve on the manifold of curvature not less than K with definite directions at the ends. Let us give an orientation to the curve γ. Then the right-hand and left-hand positions and semineighborhoods will be determined. Let us join the

ends of the curve γ by a simple geodesic open–polygon Γ in the right-hand semi-neighborhood. Let α_k be the angles between consecutive links in the polygon, and α and β be angles formed by the initial and terminal link with γ , all angles being taken from the side of the region bounded by the curves γ and Γ. The right swerve of the curve is given by

$$\lim_{\Gamma \to \gamma} (2\pi - \alpha - \beta + \sum_k (\pi - \alpha_k).$$

The left swerve is defined similarly.

The swerve has the following properties. If the point C divides the curve γ into γ_1 and γ_2, which have definite directions at C, then

$$\Psi (\gamma) = \Psi (\gamma_1) + \Psi (\gamma_2) + \pi - \vartheta,$$

where Ψ designates the swerve (left or right), and ϑ is the angle between the curves γ_1 and γ_2 at C (from the right and left).

If the manifold is Riemannian, and the curve is regular, the swerve is simply the integral of the geodesic curvature with respect to the arc length of the curve.

The right and left swerves of the curve are simply linked with the intrinsic curvature of the surface along the curve. Specifically, the sum of the right and left swerves of the curve is equal to the intrinsic curvature of the set of points on this curve.

The Gauss–Bonnet theorem can be extended to manifolds of curvature not less than K. If G is region

which is homeomorphic to a disk, and is bounded by the closed curve γ, $\omega(G)$ is the curvature of G, and $\Psi(\gamma)$ is the swerve of γ away from G, then

$$\omega(G) + \Psi(\gamma) = 2\pi.$$

Let R_1 and R_2 be two manifolds of curvature not less than K with boundary curves γ_1 and γ_2, in isometric correspondence. Let the sum of the swerves of any two corresponding pieces of γ_1 and γ_2 be nonnegative. There is then a manifold R of curvature not less than K which is the union of two subsets isometric to R_1 and R_2 whose intersection is the curve γ corresponding to γ_1 and γ_2 (the gluing theorem).

As has been pointed out above, each convex surface in elliptic space of curvature K is a manifold of curvature not less than K isometric to a convex surface in elliptic space.

The most general results in this direction have been obtained by Aleksandrov. He has proved that every complete manifold of curvature not less than K (> 0) is isometric to a closed convex surface in elliptic space of curvature K. A manifold homeomorphic to a closed disk and of curvature not less than K with a non-negative swerve in any part of its boundary is isometric to a convex cap.

The publication [4] dealt with the general question of isometric immersion of a two-dimensional Riemannian manifold in a three-dimensional one. The results obtained there, together with Aleksandrov's

theorems guarantee the existence of regular immersions in elliptic space, provided the metric of the immersed manifold is sufficiently regular, and its curvature is strictly greater than the curvature of the elliptic space.

CHAPTER III

Transformation of Congruent Figures

In this chapter we will introduce and consider some special transformations of congruent figures in elliptic space into the corresponding congruent figures in Euclidean space, and vice-versa. The results will be used to study pairs of isometric surfaces in the next chapter.

1. TRANSFORMATION OF CONGRUENT FIGURES IN ELLIPTIC SPACE TO CONGRUENT FIGURES IN EUCLIDEAN SPACE

Let R be an elliptic space with curvature $K = 1$. Let us introduce Weierstrass coordinates x_i into R. Let us remove the plane $x_0 = 0$ from R and call the remainder of the space R_0. In the region R_0 the Weierstrass coordinates can be made to conform to the extra condition

$$x_0 > 0$$

and can thereby be made competely single-valued. Now the correspondence between a point of R_0 with the Weierstrass coordinates x_i to the point in four-dimensional Euclidean space with the same Cartesian coordinates represents an isometric map of R_0 into the unit hemisphere

$$x^2 = 1, \quad x_0 > 0.$$

In the region R_0 of elliptic space let there be a figure F, which is transformed by the motion A of elliptic space to the congruent figure AF, also belonging to R_0. Let us compare each point $x \subset F$ with the point in Euclidean space $E_0 (x_0 = 0)$ by the formula

$$Tx = \frac{x - e_0(xe_0)}{e_0(x + Ax)}.$$

The point Tx certainly belongs to E_0, since $e_0 Tx = 0$.

When the point x runs through the figure F, the corresponding point Tx in Euclidean space E_0 describes a figure TF.

Lemma 1. The mapping T of the figure F in elliptic space R onto the figure TF in Euclidean space is a geodesic mapping.*

Proof. We will first demonstrate that T is a topological mapping.

Since both F and AF belong to R_0, $e_0 (x + Ax) > 0$ and, consequently, the map T is continuous. We will demonstrate that the images of different points x and y under T are different. Let us assume that Tx = Ty. Then, obviously, $y = \lambda x + \mu e_0$, and, consequently,

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* What is meant here is that T maps geodesics in $R_0 \cap A^{-1} R_0$ to geodesics in the image of this set under T. Thus one can suppose that $F = R_0 \cap A^{-1} R_0$. A similar statement applies to several of the following lemmas. —Ed.

$$Ty = \frac{x - e_0\,(xe_0)}{e_0\,(x + Ax) + \frac{\mu}{\lambda}\,e_0\,(e_0 + Ae_0)}\,.$$

Without loss of generality it can be supposed that $Ae_0 \neq - e_0$, since A and -A provide the same movement of elliptic space. Hence $e_0\,(e_0 + Ae_0) > 0$ and, consequently, the equality Ty = Tx is only possible in two cases: $\mu = 0$ and x $-e_0\,(e_0x) = 0$. In the first case y $= \lambda$ x, and since $x^2 = y^2 = 1$, and $x_0,\,y_0 > 0$, x must equal y. In the second case $x = e_0$, y = $= e_0\,(\lambda + \mu)$, from which, as in the previous case, we obtain x = y. Since, moreover, T is a continuous mapping, this proves that it is topological.

Let us now show that T is a geodesic mapping. For this it is enough to demonstrate that the inverse map T^{-1} is geodesic. Let us take any plane in E_0 . It is given by the equation

$$(ay) + b = 0,$$

where a is a vector, and b is a scalar. Substituting Tx into this equation in place of y, we obtain the equation for its image in R under T^{-1}. The equation

$$(a,\ Tx) + b = 0$$

becomes linear with respect to x after multiplication by e_0 (x + Ax), and, consequently, is the equation of plane R. Since the topological mapping T^{-1} transforms the planes in E_0 to the planes in R, it is geodesic, and T will be geodesic along with it. This proves lemma 1.

Let there be two congruent figures F and F' in the

region R_0 of elliptic space R. Let A be a motion of elliptic space which transforms F to F'. According to lemma 1, the mapping of T' of F' in Euclidean space E_0 is given by the formula

$$T'x' = \frac{x' - e_0\,(x'e_0)}{e_0\,(x' + A^{-1}x')},$$

and is a geodesic mapping.

Lemma 2. The mapping Ω of the figure TF onto the figure T'F' in Euclidean space E_0., which sends the point Tx to T'Ax, is a motion. Consequently, the figures TF and T'F' are congruent.

Proof. First of all, let us point out that the mapping Ω depends continuously on A, and that, if A is the identity map, then Ω is also the identity. Hence it is enough to show that the mapping Ω is isometric for any A.

Let x and y be two arbitrary points on F. We will demonstrate that

$$(Tx - Ty)^2 = (T'Ax - T'Ay)^2.$$

Since the formulae for Tx and T'Ax are homogeneous of degree zero with respect to x, and $e_0\,(x + Ax) > 0$ on F, the coordinates x may be normalized by the condition

$$e_0\,(x + Ax) = 1$$

and similarly for y. Then, taking x - y = z, we have:

$$(Tx - Ty)^2 = (z - e_0 (ze_0))^2 = z^2 - (ze_0)^2;$$
$$(T'Ax - T'Ay)^2 = (Az - e_0 (Az, e_0))^2 = (Az)^2 - (e_0, Az)^2.$$

But, $(Az)^2 = z^2$, and by the normalization of the coordinates, $e_0 (z + Az) = 0$. Consequently, $(ze_0)^2 = (e_0, Az)^2$, and

$$(Tx - Ty)^2 = (T'Ax - T'Ay)^2.$$

The lemma is proved.

2. TRANSFORMATION OF CONGRUENT FIGURES IN EUCLIDEAN SPACE ONTO CONGRUENT FIGURES IN ELLIPTIC SPACE

Let Φ be an arbitrary figure in Euclidean space E_0, and B be any motion in E_0. Let us make each point x in the figure Φ correspond to the point in elliptic space:

$$Sx = \rho (2x + e_0 (1 + (Bx)^2 - x^2)),$$

where ρ is a normalizing factor defined by the condition $(Sx)^2 = 1$. It is easy to see that

$$\frac{1}{\rho^2} = 1 + 2(x^2 + (Bx)^2) + (x^2 - (Bx)^2)^2 \geqslant 1.$$

Lemma 3. The mapping S of the figure Φ in Euclidean space on the figure $S\Phi$ in elliptic space R is geodesic.

<u>Proof.</u> Let us first demonstrate that S is a topological mapping. To do this it is enough to show that

S is continuous and transforms distinct points on E_0
to distinct points on R.

The continuity of the mapping of S is obvious, since
$$\frac{1}{\rho} \geqslant 1.$$

Let us show that the images of different points are
different. Let us assume the opposite and that x and y
are two different points, whose images coincide, i.e.,
that Sx = Sy. Since x and y are orthogonal to the
vector e_0, Sx = Sy implies that the vectors x and y
are parallel. Designating a unit vector parallel to
them as τ, we have:

$$x = \lambda\tau, \quad y = \mu\tau;$$
$$Sx = \rho\,(2\lambda\tau + e_0\,(1 - \lambda^2 + (Bx)^2));$$
$$Sy = \rho\,(2\mu\tau + e_0\,(1 - \mu^2 + (By)^2)).$$

Since Sy = Sy, and $\tau \perp e_0$,

$$\frac{1-\lambda^2+(Bx)^2}{\lambda} = \frac{1-\mu^2+(By)^2}{\mu}.$$

Separating B into the rotation B* and the transla-
tion C, we obtain

$$Bx = \lambda\tau^* + c, \quad By = \mu\tau^* + c, \quad \tau^* = B^*\tau.$$

And the previous equality assumes the form

$$2\tau^*c + \frac{1+c^2}{\lambda} = 2\tau^*c + \frac{1+c^2}{\mu}.$$

Hence $\lambda = \mu$, and, therefore, $x - y$, which is impos-
sible. This therefore proves that S is a topological
mapping.

Now let us demonstrate that the mapping S is geodesic. Clearly, to do this we only need to show that S^{-1} transforms the planes in R to planes in E_0.

A plane in R is given by the equation $ax = 0$. Substituting Sx in place of x, we obtain an equation for its image in E_0 under S^{-1}:

$$a(2x + e_0(1 + (Bx)^2 - x^2)) = 0.$$

It is not difficult to see that this equation is linear with respect to x. Indeed,

$$(Bx)^2 = (B^*x + c)^2 = x^2 + 2cB^*x + c^2,$$

and the equation takes the form

$$a(2x + e_0(1 + 2cB^*x + c^2)) = 0.$$

Thus, the mapping S^{-1} transforms the planes in R to those of E_0, and consequently, it is geodesic. The inverse mapping S will obviously be geodesic as well. The lemma has been proved.

Now let Φ and Φ' be congruent figures in the Euclidean space E_0, and let B be the motion of this space transferring Φ to Φ'. According to lemma 3, the mapping S' of the figure Φ' in elliptic space given by the formula

$$S'x' = \rho(2x' + e_0(1 + (B^{-1}x')^2 - x'^2)),$$

is a geodesic mapping.

Lemma 4. The mapping Ω of the figure S onto the figure $S'\Phi'$ in elliptic space, which sends Sx to $S'Bx$, is a motion. Consequently, the figures $S\Phi$ and $S'\Phi'$ are congruent.

Proof. Since the mapping Ω is a continuous function

of B and is the identity, if B is the identity, it is enough to show that it is isometric for any B. For this it is, in turn, sufficient to show that at any x and y from E_0

$$(Sx - Sy)^2 = (S'Bx - S'By)^2.$$

Let us designate:

$$\alpha(x) = 2x + e_0(1 + (Bx)^2 - x^2);$$
$$\beta(x) = 2Bx + e_0(1 - (Bx)^2 + x^2).$$

We will show that for any x and y from E_0

$$\alpha(x)\alpha(y) = \beta(x)\beta(y).$$

Since x and y are perpendicular to e_0,

$$\alpha(x)\,\alpha(y) = 4xy + (1 + (Bx)^2 - x^2)(1 + (By^2) - y^2);$$
$$\beta(x)\beta(y) = 4(Bx)(By) + (1 - (Bx)^2 + x^2)(1 - (By)^2 + y^2).$$

If the motion B is expanded into the rotation B* and the parallel translation by the vector c, we obtain

$$(Bx)^2 = x^2 + 2c(B^*x) + c^2, \quad (By)^2 = y^2 + 2c(B^*y) + c^2;$$
$$(Bx)(By) = xy + c(B^*x + B^*y) + c^2.$$

Introducing these expressions into $\alpha(x)$, $\alpha(y)$ and $\beta(x)$, $\beta(y)$, we obtain

$$\alpha(x)\,\alpha(y) = \beta(x)\beta(y).$$

From this, taking it into account that for any z, in particular for z = x, or z = y,

$$(Sz)^2 = 1, \quad (S'Bz)^2 = 1$$

$$Sz = \frac{\alpha\,(z)}{\sqrt{\overline{\alpha\,(z)^2}}}, \qquad S'Bz = \frac{\beta\,(z)}{\sqrt{\overline{\beta\,(z)^2}}},$$

we can conclude that

$$(Sx - Sy)^2 = (S'Bx - S'By)^2.$$

The lemma is completely proved.

We should point out in conclusion that the correspondence of pairs of congruent figures in elliptic and Euclidean space determined in lemmas 2 and 4, is reciprocal, i.e., if for a pair of congruent figures F and F' in elliptic space we plot a pair of congruent figures F and F' in Euclidean space on the basis of lemma 2, and then plot a pair of congruent figures in elliptic space according to lemma 4, we obtain F and F'.

3. TRANSFORMATION BY INFINITESIMAL MOTIONS

The deformation of the figure F at $t = 0$ is called an infinitesimal motion when the distance between any two of its points at the moment $t = 0$ is stationary. Since the stationary nature of the distance between the points x and y is equivalent to that of the expression $(x-y)^2$ *, the velocity field of infinitesimal motion $\zeta = dx/dt$ satisfies the equation

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*In Euclidean space, $(x-y)^2$ is square of the distance, d, between the points x and y, while in elliptic space, $(x-y)^2 = 4\sin^2 d/2$.

$$(x - y)\, (\zeta\, (x) - \zeta\, (y)) = 0.$$

Conversely, any deformation of the figure, whose velocity field satisfies this equation, is an infinitesimal motion.

Lemma 5. Let $\zeta\, (x)$ be the velocity field of an infinitesimal motion of the figure F in elliptic space R. To the point

$$Tx = \frac{x - e_0(xe_0)}{(e_0 x)}$$

in Euclidean space E_0 let there correspond the vector

$$z\,(Tx) = \frac{\zeta - e_0(e_0 \zeta)}{(e_0 x)}.$$

Then the field $z\,(Tx)$ is the velocity field of the infinitesimal motion of the figure TF in Euclidean space E_0.

<u>Proof.</u> The mapping A of the space R on itself, in which the point $u = \rho\,(x + \varepsilon\zeta)$ corresponds to $Au = \rho\,(x - \varepsilon\zeta)$, is a motion, as we will now show.

The mapping A is a continuous function of ε, and at $\varepsilon = 0$ is the identity. Thus, we now only need to show that A is isometric for any ε.

Since $x^2 = 1$, $x\zeta = 0$, the normalizing factors ρ in the expressions for u and Au are identical, that is

$$\frac{1}{\rho^2} = x^2 + \varepsilon^2\zeta^2.$$

Further, since

$$x\mathfrak{f}(x) = y\mathfrak{f}(y) = 0, \quad (x-y)(\mathfrak{f}(x) - \mathfrak{f}(y)) = 0,$$

one concludes that

$$(x + \varepsilon\zeta(x))(y + \varepsilon\zeta(y)) = (x - \varepsilon\zeta(x))(y - \varepsilon\zeta(y)).$$

If it is now taken into consideration that $u^2 = Au^2 = 1$, then we have for any u and v:

$$(u - v)^2 = (Au - Av)^2.$$

And this means that A is a motion in space R.

According to lemma 2, the transformation of Euclidean space E_0, in which its point

$$w = \frac{u - e_0(e_0 u)}{e_0(u + Au)}$$

is mapped to the point

$$Bw = \frac{Au - e_0(e_0 Au)}{e_0(Au + u)},$$

is a motion. It follows from this that

$$z = \lim_{\varepsilon \to 0} \frac{w - Bw}{2\varepsilon} = \frac{\zeta - e_0(\zeta e_0)}{(e_0 x)},$$

referred to the point

$$Tx = \frac{x - e_0(x e_0)}{e_0 x},$$

is the velocity field of an infinitesimal motion. The lemma has been proved.

Lemma 6. Let z(x) be the velocity field of an infinitesimal motion of a figure in Euclidean space E_0. To each point

$$Sx = \rho\,(x + e_0), \quad x \subset \Phi$$

in elliptic space R, let there correspond the vector

$$\zeta\,(Sx) = \frac{z - e_0\,(xz)}{\sqrt{1 + x^2}}$$

Then, the field ζ is a field of infinitesimal motion in the space R.

Proof. The mapping B of Euclidean space onto itself, in which the point $u = x + \varepsilon z$ is compared with the point $Bu = x - \varepsilon z$, is a motion. Indeed, it is continuous with respect to ε and the identity if $\varepsilon = 0$, and for any x and y,

$$(x - y \pm \varepsilon\,(z\,(x) - z\,(y))^2 = (x - y)^2 + \varepsilon^2\,(z\,(x) - z\,(y))^2,$$

i.e., for any u and v

$$(u - v)^2 = (Bu - Bv)^2.$$

According to lemma 4, the motion B of Euclidean space corresponds to motion A of elliptic space, which transforms the point

$$w = \rho\,(2u + e_0\,(1 + (Bu)^2 - u^2))$$

to the point

$$Aw = \rho\,(2Bu + e_0\,(1 - (Bu)^2 + u^2)).$$

It follows from this that

$$\zeta = \lim_{\varepsilon \to 0} \frac{w - Aw}{4\varepsilon} = \frac{z - e_0\,(xz)}{\sqrt{1 + x^2}},$$

associated with the point

$$Sx = \frac{x + e_0}{\sqrt{1 + x^2}},$$

is the velocity field of infinitesimal motion in elliptic space.

The lemma is proved.

The correspondences of figures in elliptic and Euclidean spaces and their infinitesimal motions determined by lemmas 5 and 6 are mutually reciprocal.

4. TRANSFORMATION OF STRAIGHT LINES AND PLANES

Let us assume we have two straight lines g' and g'' given by the following equations in elliptic space R:

$$x' = \rho\,(a' + \lambda\tau');$$
$$x'' = \rho\,(a'' + \lambda\tau''),$$

and that the constants are normalized in the following way:

$$a'^2 = a''^2 = 1, \quad a'\tau' = a''\tau'' = 0, \quad \tau'^2 = \tau''^2 \neq 0.$$

It is easy to check that for any λ and μ

$$(x'\,(\lambda) - x'\,(\mu))^2 = (x''\,(\lambda) - x''\,(\mu))^2,$$

and, therefore, that the mapping of the straight line g' on g", which compares points corresponding to the same values of the parameter λ, is a motion.

According to lemma 2 in this section, the equations

$$y' = \frac{x' - e_0(x'e_0)}{e_0\,(x' + x'')}\,,$$

$$y'' = \frac{x'' - e_0(x''e_0)}{e_0\,(x' + x'')}$$

in Euclidean space E_0 give two congruent figures h' and h". According to lemma 1, they are straight lines.

Since the normalizing factors ρ in the equations for the lines g' and g" are identical, the right-hand sides of the equations for the lines h' and h" are linear fractional expressions in λ.

The mapping of the line g' on the line h' is a homeomorphism by virtue of the correspondence of the parameters λ. Hence y' cannot vanish identically. And since y'(λ) is a linear fractional function of λ, y'$_\lambda \neq 0$ for all λ. The same conclusion can be drawn with regard to y".

According to lemma 2, the mapping of the line h' on the line h", which compares points having the same values of the parameter λ, is a motion. Hence it follows that

$$|\,y'_\lambda\,| = |\,y''_\lambda\,|.$$

Now let us assume we have in Euclidean space E_0 two straight lines h' and h" given by the equations

$$y' = b' + \mu t';$$
$$y'' = b'' + \mu t'',$$

and that $|t'| = |t''| \neq 0$. Clearly, the mapping of the line h' on the line h'', which compares equal values of the parameters μ, is a motion.

According to lemma 4, the equations

$$x' = \rho \left(2y' + e_0 \left(1 - y'^2 + y''^2\right)\right),$$
$$x'' = \rho \left(2y'' + e_0 \left(1 - y''^2 + y'^2\right)\right)$$

in elliptic space R give two congruent figures g' and g''. According to lemma 3, they must be straight lines.

Since the expressions

$$2y' + e_0 \left(1 - y'^2 + y''^2\right),$$
$$2y'' + e_0 \left(1 - y''^2 + y'^2\right)$$

are linear in μ, and the coefficients are different from zero at μ, the derivatives x'_μ and x''_μ do not vanish anywhere.

Since the mapping of g' on g'', which compares equal values of the parameters μ, is a motion,

$$|x'_\mu| = |x''_\mu|.$$

We can draw similar conclusions for planes.

Let us assume we have two planes α' and α'' given by the following equations in elliptic space

$$x' = \rho \left(a' + \lambda \tau'_1 + \mu \tau'_2\right);$$
$$x'' = \rho \left(a'' + \lambda \tau''_1 + \mu \tau''_2\right),$$

and that the constants are normalized as follows:

$$a'^2 = a''^2 = 1, \quad a'\tau_1' = a''\tau_1'', \quad a'\tau_2' = a''\tau_2'',$$

$$\tau_1'^2 = \tau_1''^2, \quad \tau_2'^2 = \tau_2''^2, \quad \tau_1'\tau_2' = \tau_1''\tau_2'', \quad (a'\tau_1'\tau_2') \neq 0, \ (a''\tau_1''\tau_2'') \neq 0.$$

It is easy to see that the mapping of the plane α' on α'', which compares equal values of the parameters λ and μ is a motion.

According to lemmas 1 and 2, the equations

$$y' = \frac{x' - e_0(x'e_0)}{e_0\,(x' + x'')},$$

$$y'' = \frac{x'' - e_0(x''e_0)}{e_0\,(x' + x'')}$$

give two planes — β' and β'' — in Euclidean space and the mapping of one plane on the other, which compares equal values of the parameters λ and μ, is a motion.

Since y' (λ, μ) is a linear fractional expression in λ and μ, and the mapping of the plane α' on the plane β' is a homeomorphism, the vectors y_λ' and y_μ' are independent and different from zero for any λ and μ. The same thing can be said of the vectors y_λ'' and y_μ'' .

Since the correspondence between the planes β' and β'', which compares equal values of the parameters λ and μ, is isometric (lemma 2),

$$y_\lambda'^2 = y_\lambda''^2, \quad y_\lambda' y_\mu' = y_\lambda'' y_\mu'', \quad y_\mu'^2 = y_\mu''^2.$$

Now let us assume we have two planes β' and β''

given by the following equations in Euclidean space E_0

$$y' = b' + \lambda t_1' + \mu t_2';$$
$$y'' = b'' + \lambda t_1'' + \mu t_2'',$$

and that

$$t_1'^2 = t_1''^2, \quad t_2'^2 = t_2''^2, \quad t_1' t_2' = t_1'' t_2'',$$
$$t_1'^2 t_2'^2 - (t_1' t_2')^2 \neq 0.$$

Clearly the mapping of the plane β' on β'', which compares equal values of the parameters λ and μ, is a motion. According to lemmas 3 and 4, the equations

$$x' = \rho \, (2y' + e_0 \, (1 - y'^2 + y''^2)),$$
$$x'' = \rho \, (2y'' + e_0 \, (1 - y''^2 + y'^2))$$

in elliptic space R give two planes α' and α'', the correspondence between the points of which, when equal values of the parameters λ and μ are compared is a motion.

It is easy to see that the expressions

$$2y' + e_0 \, (1 - y'^2 + y''^2)$$
$$2y'' + e_0 \, (1 - y''^2 + y'^2)$$

are linear with respect to λ and μ, and that the coefficients at λ and μ in each of the expressions are independent by virtue of the independence of the vectors τ_1', τ_2' and τ_1'', τ_2'', respectively. It follows from this that

$$(x' \, x_\lambda' \, x_\mu') \neq 0, \quad (x'' x_\lambda'' x_\mu'') \neq 0.$$

Since the correspondence between the points on the planes α' and α'' , which compares equal values of the parameters λ, and μ , is isometric, we have the equalities

$$x_\lambda'^2 = x_\lambda''^2, \quad x_\lambda' \, x_\mu' = x_\lambda'' x_\mu'', \quad x_\mu'^2 = x_\mu''^2$$

CHAPTER IV

Isometric Surfaces

The method we have just considered for comparing each pair of congruent figures in elliptic space with a pair of congruent figures in Euclidean space, and vice-versa, may easily be extended to cover the case of infinitesimally congruent figures, in particular, to isometric surfaces. The present section deals with this problem.

1. TRANSFORMATION OF ISOMETRIC SURFACES

Let there be two smooth* isometric surfaces F' and F'' in the region R_0 ($x_0 > 0$) of elliptic space R with a curvature K = 1. Let us introduce the coordinate system u, v in such a way that isometrically-corresponding points have the same values of the parameters u, v. Let

$$x = x' \, (u, v), \quad x = x'' \, (u, v)$$

be equations for these surfaces, and let

$$(x' x'_u x'_v) \neq 0 \text{ and } (x'' x''_u x''_v) \neq 0.$$

Let us consider the two surfaces Φ' and Φ'' given

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*That is, surfaces which have a tangent plane at every point — Ed.

by the following equations in Euclidean space $E_0 (x_0 = 0)$

$$y = y' (u, v), \quad y = y'' (u, v),$$

where

$$y' = \frac{x' - e_0 (x' e_0)}{e_0 (x' + x'')}, \quad y'' = \frac{x'' - e_0 (x'' e_0)}{e_0 (x' + x'')}.$$

The following theorem is valid:

Theorem 1. The surfaces Φ' and Φ'' are smooth, do not have any singularities* and are isometric. They are congruent if, and only if, the surfaces F' and F'' are congruent.

Proof. Let us take two isometrically–corresponding points A' and A'' on the surfaces F' and F''. Without loss of generality it can be supposed that they correspond to the parameters u = v = 0. Let us draw tangent planes α' and α'', respectively at the points A' and A''. These planes can be given by the equations:

$$x = \bar{x}' (u, v), \quad x = \bar{x}''(u, v),$$

where

$$\bar{x}' = \rho (x' (0, 0) + ux'_u (0, 0) + vx'_v (0, 0));$$
$$\bar{x}'' = \rho (x'' (0, 0) + ux''_u (0, 0) + vx''_v (0, 0)).$$

The vectors in the right–hand sides of the expressions $\bar{x}'$ and $\bar{x}''$, clearly satisfy the equations:

$$x'^2 = x''^2 = 1, \quad x'x'_u = x'x'_v = 0, \quad x''x''_u = x''x''_v = 0,$$
$$x'^2_u = x''^2_u, \, x'_u x'_v = x''_u x''_v, \, x'^2_v = x''^2_v.$$

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*That is, the immersion is of rank two everywhere — Ed.

Hence, according to par. 4 in Chapter III, the equations

$$y = \bar{y}'\,(u,\ v), \quad y = \bar{y}''\,(u,\ v),$$

in which

$$\bar{y}' = \frac{\bar{x}' - e_0(\bar{x}''e_0)}{e_0\,(\bar{x}' + \bar{x}'')}, \quad \bar{y}'' = \frac{\bar{x}'' - e_0\,(\bar{x}''e_0)}{e_0\,(\bar{x}' + \bar{x}'')},$$

in Euclidean space E_0 give the two planes β' and β''. The mapping of the plane α^i on the plane β^i, in which the point (u, v) of the plane α^i is compared with the point β^i with the same coordinates u, v, is a non-degenerate projective (geodesic) mapping (Chap. 3). The mapping of the plane α' on the plane α'', in which equal values of the parameters u, v are compared, is a motion.

Since the distance between the corresponding points on the surface F^i and its tangent plane α^i near point A^i is of the order of at least $u^2 + v^2$, the distance between the corresponding points on the plane β^i and Φ^i near the point B^i (corresponding to A^i) is of the order of not less than $u^2 + v^2$.

It follows from this that each figure Φ^i is really a surface (without singularities), that the plane β^i is a tangent plane for the surface Φ^i, and that the mapping of the surface Φ^i on the plane β^i, which compares equal values of the parameters u, v, is isometric at the point B^i.

Since the mapping of the plane β' on. β'', which compares equal values of the parameters u, v, is also isometric, that of the surface Φ' on Φ'', in which points with identical coordinates u, v are compared, is isometric at u = v = 0. But the point u = v = 0 was taken completely arbitrarily. Hence, the surfaces Φ' and Φ'' are isometric.

As is known (Chap. 3), the congruence of the figures F' and F'' results in the congruence of the figures Φ' and Φ'', and, vice-versa.

The theorem is completely proved.

Let there be two smooth isometric surfaces Φ' and Φ'' in the Euclidean space E_0 and let them be parametrized in such a way that the isometrically-corresponding points have the same coordinates u, v. Let

$$y = y'\,(u,\ v),\ \ y = y''\,(u,\ v)$$

be equations for these surfaces, and let

$$y'_u \times y'_v \neq 0,\ \ y''_u \times y''_v \neq 0.$$

Let us consider in elliptic space R two surfaces F' and F'' given by the equations:

$$x = x'\,(u,\ v),\ \ x = x''\,(u,\ v),$$

in which

$$x' = \rho\,(2y' + e_0\,(1 - y'^2 + y''^2)),$$
$$x'' = \rho\,(2y'' + e_0\,(1 - y''^2 + y'^2)).$$

Theorem 2. The surfaces F' and F'' are smooth, do not have any singularities and are isometric. They are congruent if, and only if, the surfaces Φ' and Φ'' are congruent.

The proof of this theorem is similar to that for Theorem 1. Hence we will not give it here.

We should point out in conclusion that if for a pair of isometric surfaces F' and F'' in elliptic space we plot a pair of corresponding isometric surfaces Φ' and Φ'' in Euclidean space (Theorem 1), and then use these surfaces to plot a pair of isometric surfaces in elliptic space (Theorem 2), we obtain F' and F''.

2. TRANSFORMATION OF LOCALLY CONVEX ISO-METRIC SURFACES IN ELLIPTIC SPACE

According to Theorem 1, each pair of isometric surfaces F' and F'' in the region R_0 of elliptic space R is compared with a pair of isometric surfaces Φ' and Φ'' in Euclidean space E_0. In certain cases it happens that the convexity of the surfaces F' and F'' guarantees the convexity of the surfaces Φ' and Φ''. In fact, the following theorem is valid.

Theorem 3. If the surfaces F' and F'' in elliptic space, which were mentioned in Theorem 1, are locally convex and compatibly oriented, both the surfaces F' and F'' being visible from inside at the point $e_0 = (1, 0, 0, 0)*$, then the corresponding surfaces Φ' and Φ'' in Euclidean space are also locally convex.

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*That is, a line segment in R_0 beginning at P_0 intersects F' or F'' in at most one point. — Ed.

The expression "surface is locally convex" is used in the sense that a sufficiently small neighborhood of each point on the surface is a convex surface. For the surface to be locally convex, it is enough for the second fundamental form not to have values of different signs.

Proof of Theorem 3.

Let us take an arbitrary point y' and a point y' + Δy' close to it on the surface Φ'. Suppose that the corresponding points on the surfaces F' and F'' are x', x' + Δx', and x'', x'' + Δx'', respectively. Let us join the points x' and x' + Δx' by the segment γ' on the surface F', and the point x'' and x'' + Δx'' by the segment γ'' on the surface F''.

Then, according to para. 3 in Chapter I,

$$x' + \Delta x' = \rho \left(\left(1 - \frac{\Delta s^2}{2} \right) x' + \Delta s \tau' + \frac{\Delta s^2}{2} k' \nu' + \ldots \right),$$

$$x'' + \Delta x'' = \rho \left(\left(1 - \frac{\Delta s^2}{2} \right) x'' + \Delta s \tau'' + \frac{\Delta s^2}{2} k'' \nu'' + \ldots \right),$$

where τ' and τ'' are the unit tangent vectors of the geodesics γ' and γ'' at the points x' and x''; ν' and ν'' are the unit vectors of the principal normals of these curves; k' and k'' are the curvatures of the curves γ' and γ'', and Δs is the distance between the points x^i and $x^i + \Delta x^i$ on the surface F^i. Terms of the order higher than Δs^2 have not been included. Let us substitute these expressions into the formula for y'. We

then obtain*

$$y' + \Delta y' = \frac{x' + \varepsilon' - e_0(e_0 x' + e_0 \varepsilon')}{e_0 x' + e_0 x'' + (e_0 \varepsilon' + e_0 \varepsilon'')} + 0\,(\Delta s^3),$$

where for the sake of brevity we designate

$$\varepsilon' = \Delta s \tau' + \frac{\Delta s^2}{2} k' \nu', \quad \varepsilon'' = \Delta s \tau'' + \frac{\Delta s^2}{2} k'' \nu''.$$

Separating the principal term in the expression $y' + \Delta y'$ up to the order Δs^2, we obtain

$$\Delta y' = \frac{\Delta s}{e_0\,(x' + x'')} \{ - y'\,(e_0 \tau' + e_0 \tau'') + \tau' - e_0\,(e_0 \tau') \} +$$

$$+ \Delta s^2 \frac{e_0\,(\tau' + \tau'')}{(e_0(x' + x''))^2} \{ y'\,(e_0 \tau' + e_0 \tau'') - \tau' + e_0\,(e_0 \tau') \} +$$

$$+ \frac{\Delta s^2}{2 e_0\,(x' + x'')} \{ - (k' e_0 \nu' + k'' e_0 \nu'')\,y' + k' \nu' - k'' e_0\,(e_0 \nu') \} + 0\,(\Delta s^3).$$

Let us multiply $\Delta y'$ by the unit vector n' of the normal of the surface Φ' at the point y'. We should then obtain a quantity of order not less then Δs^2. It follows from this that

$$n'\,\{ - y'\,(e_0 \tau' + e_0 \tau'') + \tau' - e_0\,(e_0 \tau') \} = 0.$$

And for $n'\,\Delta y'$ we obtain

$$n'\Delta y' = \frac{\Delta s^2 k'}{2 e_0\,(x' + x'')}\,(- y'\,(e_0 \nu') + \nu')\,n' -$$

$$- \frac{\Delta s^2 \bar{k}''}{2 e_0\,(x' + x'')}\,(e_0 \nu'')\,(y' n') + 0\,(\Delta s^3).$$

Since both the expressions

$$\frac{\Delta s^2 k'}{2 e_0\,(x' + x'')} \quad \text{and} \quad \frac{\Delta s^2 k''}{2 e_0\,(x' + x'')}$$

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*Note that the terms $\underline{\Delta s^2 \underline{x'}}{2}$ and $\underline{\Delta s^2 \underline{x''}}{2}$ do not contribute to terms of order Δs^2 or less in $y' + \Delta y'$. — Ed.

are positive, it is sufficient to show for local convexity of the surface Φ' that the expressions

$$(- y' \, (e_0 \nu') + \nu') \, n' \text{ and} - (e_0 \nu'') \, (y'n') \qquad (*)$$

have the same sign, regardless of the choice of the point y' and the point near to it $y' + \Delta y'$ on the surface Φ'.

The proof that these expressions always have the same sign involves some computation. Hence we single it out with a special lemma, which will be given in the next paragraph

Let α' and α'' be two planes in elliptic space R which compared isometrically, and β' and β'' be the corresponding planes in Euclidean space E_0 (para 4, Chapter III). Let us use a_0 and b_0 to designate two corresponding points on the planes α' and α'', and a_3 and b_3 to be the corresponding unit normals at these points, and, finally c_0 and c_3 to be the point and normal of the plane β', respectively.

Lemma 1. If both planes α' and α'' in the region R_0 have the same side turned towards the point e_0, then the expressions

$$A = c_3 \, (- (e_0 \, a_3) \, c_0 + a_3), \quad B = - (e_0 b_3) \, (c_0 c_3)$$

have the same sign. During motion of the planes α'

and α'' , the signs do not change*.

The isometric correspondence between the surfaces F' and F'' naturally induces isometric correspondence in their tangent planes α' and α'' at the points x' and x''. And the fact that the expressions (*) have the same signs follows from Lemma 1.

The local convexity of the surface Φ'' is established in the same way.

The theorem is proved.

3. PROOF OF LEMMA 1.

Let us use a_1 and a_2 to designate unit orthogonal vectors at the point a_0 in the plane α', b_1 and b_2 to be the corresponding unit vectors in the plane α'', and c_1 and c_2 to be the corresponding vectors in the plane β'. Then

$$a_3 = (a_0 a_1 a_2), \quad b_3 = (b_0 b_1 b_2), \quad c_3 = (c_0 c_1 c_2).$$

In order to find an expression for the vectors c_1 and c_2, let us turn to the equation for the plane β'.

$$y = \frac{x_1 - e_0(x_1 e_0)}{e_0 (x_1 + x_2)},$$

where x_1 and x_2 are the isometrically corresponding

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* The correspondence of the sides of planes is achieved in the usual way through the isometric correspondence established between them.

points on the planes α' and α''. By differentiating y
with respect to directions corresponding to c_1 and
c_2 at the point c_0 we obtain

$$\dot{c}_1 = \frac{1}{\lambda_0^2} (a_1\lambda_0 - a_0\lambda_1) + e_0 \; (*);$$

$$c_2 = \frac{1}{\lambda_0^2} (a_2\lambda_0 - a_0\lambda_2) + e_0 \; (*),$$

in which for the sake of brevity, the notation

$$\lambda_0 = e_0 (a_0 + b_0), \quad \lambda_1 = e_0 (a_1 + b_1), \quad \lambda_2 = e_0 (a_2 + b_2)$$

is used. Substituting these expressions for c_1 and
c_2 into $c_3 = (e_0 c_1 c_2)$, we obtain

$$c_3 = \frac{1}{\lambda_0^4} (e_0, \; a_1\lambda_0 - a_0\lambda_1, \; a_2\lambda_0 - a_0\lambda_2) =$$

$$= \frac{1}{\lambda_0^3} \{\lambda_0 (e_0 a_1 a_2) - \lambda_1 (e_0 a_0 a_2) - \lambda_2 (e_0 a_1 a_0)\}.$$

Now let us turn to the expressions A and B. We
have:

$$B = - (e_0 b_3) (c_0 c_3),$$
$$(e_0 b_3) = (e_0 b_0 b_1 b_2).$$

Since

$$c_0 = \frac{a_0}{\lambda_0} + e_0 \; (*),$$

it follows

$$(c_0 c_3) = -\frac{1}{\lambda_0^3} (e_0 a_0 a_1 a_2).$$

By the conditions of the lemma, both planes α' and α'' are visible from the point e_0 from the same side. Analytically this means that the expressions $(e_0 b_0 b_1 b_2)$, and $(e_0 a_0 a_1 a_2)$ have the same sign. It follows from this that

$$B = \frac{1}{\lambda_0^3} (e_0 a_0 a_1 a_2) (e_0 b_0 b_1 b_2) > 0.$$

Let us now consider the expression

$$A = c_3 (a_3 - (e_0 a_3) c_0).$$

Firstly,

$$- (e_0 a_3) (c_0 c_3) = \frac{1}{\lambda_0^3} (e_0 a_0 a_1 a_2)^2.$$

Further,

$$(c_3 a_3) = \frac{1}{\lambda_0^3} \{ \lambda_0 (e_0 a_1 a_2) (a_0 a_1 a_2) - \lambda_1 (e_0 a_0 a_2) (a_0 a_1 a_2) -$$
$$- \lambda_2 (e_0 a_1 a_0) (a_0 a_1 a_2) \}.$$

By applying the vector identities in Chapter I, para. 1 to each of the scalar products in the braces, we have

$$(a_0 a_1 a_2) (e_0 a_1 a_2) = (e_0 a_0);$$
$$(a_0 a_1 a_2) (e_0 a_0 a_2) = - (e_0 a_1);$$
$$(a_0 a_1 a_2) (e_0 a_1 a_0) = - (e_0 a_2).$$

Thus,

$$(c_3 a_3) = \frac{1}{\lambda_0^3} \left(\lambda_0 (e_0 a_0) + \lambda_1 (e_0 a_1) + \lambda_2 (e_0 a_2) \right).$$

Introducing here the expressions for λ_0, λ_1, and λ_2, we obtain

$$(c_3 a_3) = \frac{1}{\lambda_0^3} \{ (e_0 a_0)^2 + (e_0 a_1)^2 + (e_0 a_2)^2 +$$

$$+ (e_0 a_0)(e_0 b_0) + (e_0 a_1)(e_0 b_1) + (e_0 a_2)(e_0 b_2) \}.$$

Hence, observing that

$$2 | (e_0 a_i)(e_0 b_i) | \leqslant (e_0 a_i)^2 + (e_0 b_i)^2,$$

we obtain

$$(c_3 a_3) \geqslant \frac{1}{2\lambda_0^3} \{ (e_0 a_0)^2 + (e_0 a_1)^2 + (e_0 a_2)^2 -$$

$$- (e_0 b_0)^2 - (e_0 b_1)^2 - (e_0 b_2)^2 \}.$$

And since

$$(e_0 a_0)^2 + (e_0 a_1)^2 + (e_0 a_2)^2 + (e_0 a_3)^2 =$$
$$= (e_0 b_0)^2 + (e_0 b_1)^2 + (e_0 b_2)^2 + (e_0 b_3)^2,$$

therefore

$$(c_3 a_3) \geqslant \frac{1}{2\lambda_0^3} ((e_0 b_3)^2 - (e_0 a_3)^2) =$$

$$= \frac{1}{2\lambda_0^3} ((e_0 b_0 b_1 b_2)^2 - (e_0 a_0 a_1 a_2)^2).$$

Taking into account the expressions for $-(e_0 a_3) (c_0 c_3)$

and the lower bound for $(c_3 a_3)$, we obtain

$$A \geqslant \frac{1}{2\lambda_0^3} \left((e_0 b_0 b_1 b_2)^2 + (e_0 a_0 a_1 a_2)^2 \right) > 0.$$

The lemma is completely proved.

4. TRANSFORMATION OF LOCALLY CONVEX ISO-METRIC SURFACES IN EUCLIDEAN SPACE.

According to Theorem 2 of this chapter, each pair of isometric surfaces Φ' and Φ'' in Euclidean space E_0 corresponds to a pair of isometric surfaces F' and F'' in elliptic space R. This result may be complemented by the following theorem.

***Theorem* 4.** If the surfaces Φ' and Φ'' in Euclidean space E, mentioned in Theorem 2, are locally convex, oppositely oriented and visible from inside at the point (0, 0, 0, 0), the corresponding surfaces F' and F'' in elliptic space are locally convex.

<u>Proof</u>. Let us take the arbitrary point x' and point $x' + \Delta x'$ close to it on the surface F'. The corresponding points on the surfaces Φ' and Φ'' are the points y', y' + Δy', and y'', y'' + Δy''. Let us join points y' and y' + Δy' by the segment γ' on the surface Φ', and the points y'' and y'' + Δy'' with the corresponding segment γ'' on the surface Φ''. We have:

$$\Delta y' = \Delta s \tau' + \frac{\Delta s^2}{2} k' \nu' + \ldots,$$

$$\Delta y'' = \Delta s \tau'' + \frac{\Delta s^2}{2} k'' \nu'' + \ldots,$$

where τ' and τ'' are unit vectors of the tangent

geodesics γ' and γ'' at the points y' and y'', respectively, k' and k'' are curvatures, and v' and v'' are the principal normals of these curves.

Taking into account the expression for x' in terms of y' and y''

$$x' = \rho \ (2y' + e_0 \ (1 - y'^2 + y''^2)),$$

we obtain the following expression for Δx':

$$\Delta x' = \frac{\Delta \rho}{\rho} x' + \rho \Delta \Omega + \Delta \rho \Delta \Omega,$$

where

$$\Delta \Omega = 2\Delta s \ \{\tau' + e_0 \ (- y'\tau' + y''\tau'')\} +$$
$$+ \ \Delta s^2 \ \{k'v' + e_0 \ (- k'v'y' + k''v''y'')\} + 0 \ (\Delta s^3).$$

Let us multiply Δ x' by the unit vector n' of the normal of the surface F' at the point x'. Since n'x' = 0, and Δ x'n' must be of an order not lower than Δ s^2,

$$n' \ (\tau' + e_0 \ (- y'\tau' + y''\tau'')) = 0$$

and, consequently,

$$\Delta x' n' = \Delta s^2 k' \ (n'v' - (n'e_0) \ (v'y')) + \Delta s^2 k'' \ (n'e_0) \ (v''y'') + \ldots .$$

Terms of the order higher than Δ s^2 have not been included.

In order to complete the proof of the theorem, i.e., to establish the local convexity of the surface F', we need only demonstrate that the expressions

$$n'v' - (e_0 n') \ (v'y') \text{and} \ (e_0 n') \ (v''y'')$$

have the same sign. Just as in the proof of the previous theorem, this involves computing, which can be conveniently done with more symmetrical symbols for the vectors. In view of this we will formulate another lemma for isometric planes, the proof of which is given below.

Let us assume we have two planes β' and β'', which correspond isometrically, in Euclidean space E_0. According to para. 4 of Chapter III, the induced correspondence between the planes α' and α'' in elliptic space is also isometric.

Lemma 2. Let a_0 and b_0 be two isometrically corresponding points on the planes β' and β'', let a_3 and b_3 be corresponding unit normals of the planes at these points, and let c_0 and c_3 be the point and normal of plane α', respectively. Then, if both planes β' and β'' have opposite sides turned towards the origin of the coordinates, the expressions

$$A = (c_3 a_3) - (e_0 c_3) \ (a_0 a_3) \text{and} \ B = (e_0 c_3) \ (b_0 b_3)$$

differ from zero and have the same sign.

Using this lemma it is easy to complete the proof of Theorem 4. Indeed, the isometric correspondence of the surfaces Φ' and Φ'' induces isometric correspondence of their tangent planes β' and β'' at the points y' and y''. The planes in elliptic space corresponding to β' and β'' are tangent planes of the surfaces F' and F''. In fact, the plane α' touches F' at

the point x'. The expressions

$$n'v' - (e_0 n') \ (v'y') \text{ and } (e_0 n') \ (v''y'')$$

are just the quantities A and B, mentioned in the lemma, and, consequently, have the same sign. Thus, the surface F' is locally convex. The convexity of the surface F'' is established in a similar way.

The theorem is proved.

5. PROOF OF LEMMA 2.

Let us use a_1 and a_2 to designate unit orthogonal vectors at the point a_0 of the plane β', b_1 and b_2 to designate the isometrically corresponding unit vectors at the plane β'', and c_1 and c_2 to be the corresponding vectors in the plane α'. Then

$$a_3 = (e_0 a_1 a_2), \quad b_3 = (e_0 b_1 b_2), \quad c_3 = (c_0 c_1 c_2).$$

The equation for the plane α' is

$$x' = \rho \ (2y' + e_0 \ (1 - y'^2 + y''^2)).$$

Differentiating x' with respect to the corresponding directions, we obtain the following expressions for c_1 and c_2:

$$c_1 = \frac{\rho'}{\rho} c_0 + 2\rho \ (a_1 + e_0 \ (- a_0 a_1 + b_0 b_1));$$

$$c_2 = \frac{\rho'}{\rho} c_0 + 2\rho \ (a_2 + e_0 \ (- a_0 a_2 + b_0 b_2)).$$

Hence,

$$c_3 = 4\rho^2 \left(c_0, \; a_1 + e_0 \left(- a_0 a_1 + b_0 b_1 \right), \; a_2 + e_0 \left(- a_0 a_2 + b_0 b_2 \right) \right).$$

Let us consider the expression

$$B = (e_0 c_3) \; (b_0 b_3).$$

Taking into account the expression obtained for c_3 and

$$c_0 = \rho \left(2a_0 + e_0 \left(1 - a_0^2 + b_0^2 \right) \right),$$

we obtain

$$(\bar{e}_0 c_3) = 8\rho^3 \left(a_0 a_1 a_2 e_0 \right) = - 8\rho^3 \left(a_0 a_3 \right)$$

and, consequently,

$$B = - 8\rho^3 \left(a_0 a_3 \right) \left(b_0 b_3 \right).$$

Since the planes β' and β'' have their opposite sides turned to the origin of the coordinates, the expressions $(a_0 a_3)$ and $(b_0 b_3)$ have different signs. Hence,

$$B > 0.$$

Let us now consider the expression A.

$$A = (a_3 c_3) - (e_0 c_3) \; (a_0 a_3).$$

We have

$$- (\bar{e}_0 c_3) \; (a_0 a_3) = 8\rho^3 \left(e_0 a_0 a_1 a_2 \right)^2.$$

Applying the vector identity in Chapter I, para. 1, to

the right-hand side of this equality, we obtain

$$(e_0 c_3) (a_0 a_3) = 8\rho^3 (a_0{}^2 - (a_0 a_1)^2 - (a_0 a_2)^2).$$

Similarly,

$$(a_3 c_3) = 4\rho^3 (e_0 a_1 a_2) (2a_0 + e_0 (1 - a_0{}^2 + b_0{}^2)),$$
$$a_1 + e_0 (- a_0 a_1 + b_0 b_1), \; a_2 + e_0 (- a_0 a_2 + b_0 b_2) =$$
$$4\rho^3 \{1 - a_0^2 + b_0^2 + 2 (a_0 a_1)^2 + 2 (a_0 a_2)^2 -$$
$$- 2 (a_0 a_1) (b_0 b_1) - 2 (a_0 a_2) (b_0 b_2)\}.$$

and the following expression is obtained for A:

$$A = 4\rho^3 (1 + a_0^2 + b_0^2 - 2 (a_0 a_1) (b_0 b_1) - 2 (a_0 a_2) (b_0 b_2)).$$

Taking it into account that

$$2 | (a_0 a_1) (b_0 b_1) | \leqslant (a_0 a_1)^2 + (b_0 b_1)^2,$$
$$2 | (a_0 a_2) (b_0 b_2) | \leqslant (a_0 a_2)^2 + (b_0 b_2)^2,$$

we obtain

$$A \geqslant 4\rho^3 (1 + a_0^2 + b_0^2 - (a_0 a_1)^2 - (a_0 a_2)^2 - (b_0 b_1)^2 - (b_0 b_2)^2).$$

And since

$$a_0^2 = (a_0 a_1)^2 + (a_0 a_2)^2 + (a_0 a_3)^2,$$
$$b_0^2 = (b_0 b_1)^2 + (b_0 b_2)^2 + (b_0 b_3)^2,$$

it follows that

$$A \geqslant 4\rho^3 (1 + (a_0 a_3)^2 + (b_0 b_3)^2) > 0.$$

The lemma is completely proved.

CHAPTER V

Infinitesimal Deformations of Surfaces in Elliptic Space

In this section we will establish a one-to-one correspondence between surfaces in Euclidean and elliptic space and their infinitely small deformations. Thus, the problem of infinitesimal deformations of a surface in elliptic space can be reduced to infinitesimal deformations of the corresponding surface in Euclidean space.

1. PAIRS OF ISOMETRIC SURFACES AND INFINITESIMAL DEFORMATIONS

Let the surface F in elliptic space R be subjected to an infinitesimal deformation, and let τ be its field of velocities. This deformation is termed an infinitesimal deformation when the lengths of the curves on the surface are stationary. The stationary nature of the lengths of curves on the surface implies the stationary nature of its line element. Hence, as in Euclidean space, we obtain an equation for an infinitesimal deformation by differentiating the line element with respect to the deformation parameter

$$\frac{\delta}{\delta t}\, dx^2 = 0,$$

or

$$dx\, d\tau_{\prime} = 0,$$

where $\tau = \dfrac{\delta x}{\delta \tau}$ is the velocity field of the deformation. The equation for infinitesimal deformations of a surface in Euclidean space in Cartesian coordinates is exactly the same.

Let us assume we have two close isometric surfaces Φ_1 and Φ_2 in Euclidean space given by the equations in Cartesian coordinates:

$$y_1 = y_1\,(u,\,v),\quad y_2 = y_2\,(u,\,v),$$

and that the isometrically corresponding points have the same parameter values, u, v.

It is well known that the vector field

$$z = y_1 - y_2$$

is a field of an infinitesimal deformation of the surface Φ, which is given by

$$y = y_1 + y_2$$

Then

$$dy\, dz = dy_1^2 - dy_2^2 = 0$$

because of the equality of the line elements of the surfaces Φ_1 and Φ_2.

Now let us assume we have the surface Φ:

$$y = y\,(u, v)$$

and z is the field of an infinitely small deformation of it. The equations

$$y_1 = y + \lambda z, \quad y_2 = y - \lambda z$$

for a sufficiently small λ then give the two isometric surfaces, since

$$dy_1^2 = dy^2 + \lambda^2\,dz^2, \quad dy_2^2 = dy^2 + \lambda^2\,dz^2.$$

Both these results can be extended to cover the case of surfaces in elliptic space. That is, let us assume we have two close isometric surfaces F_1 and F_2, in elliptic space, given by the equations in Weierstrass coordinates:

$$x = x_1\,(u, v), \quad x = x_2\,(u, v),$$

and that the isometric correspondence between the surfaces is achieved by the equality of parameters. Then, for the surface F:

$$x = \rho\,(x_1 + x_2)$$

the vector field

$$\varsigma = \rho\,(x_1 - x_2)$$

is the field of an infinitesimal deformation.

Indeed,

$$dx = d\rho \left(x_1 + x_2\right) + \rho \left(dx_1 + dx_2\right);$$
$$d\varsigma = d\rho \left(x_1 - x_2\right) + \rho \left(dx_1 - dx_2\right).$$

Hence, taking it into account that

$$x_1^2 = x_2^2 = 1, \quad x_1\,dx_1 = x_2\,dx_2 = 0, \quad dx_1^2 = dx_2^2,$$

we obtain

$$dx \cdot d\varsigma = 0,$$

That is to say the field ς is the field of an infinitesimal deformation of the surface **F**.

Now let **F**:

$$x = x\,(u, v)$$

be a surface in elliptic space, and ς be the field of an infinitesimal deformation of it. Then for small values of λ the equations

$$x_1 = \rho \left(x + \lambda\varsigma\right), \quad x_2 = \rho \left(x - \lambda\varsigma\right)$$

give the two isometric surfaces $\mathbf{F}_1$ and $\mathbf{F}_2$. Indeed,

$$dx_1 = d\rho \left(x + \lambda\varsigma\right) + \rho \left(dx + \lambda\,d\varsigma\right),$$
$$dx_2 = d\rho \left(x - \lambda\varsigma\right) + \rho \left(dx - \lambda\,d\varsigma\right),$$

Taking it into account that

$$x\varsigma = 0, \quad d\left(x\varsigma\right) = x\,d\varsigma + \varsigma\,dx = 0, \quad dx\,d\varsigma = 0,$$

we obtain

$$dx_1^2 = dx_2^2,$$

i.e., the surfaces F_1 and F_2 are isometric.

2. TRANSFORMATION OF SURFACES AND THEIR INFINITESIMAL DEFORMATIONS

Since the problem of a field of infinitesimal deformation of a surface is essentially the problem of an infinitely close isometric surface, and since the correspondence between pairs of isometric surfaces in elliptic and in Euclidean space has been established in Chapter IV, a correspondence can also be established between surfaces and their infinitesimal deformations in elliptic and Euclidean space. In fact, the following two theorems are valid:

Theorem 1. If ζ is the field of an infinitesimal deformation of the surface F: $x = x$ (u, v) in elliptic space R, then

$$z = \frac{\zeta - e_0 (\varsigma e_0)}{(e_0 x)}$$

is the field of an infinitesimal deformation of the surface φ:

$$y = \frac{x - e_0(x e_0)}{(e_0 x)}$$

in Euclidean space E_0. The field z is trivial if, and only if, the field ζ is trivial.

Theorem 2. If z is the field of an infinitesimal deformation of the surface Φ: y = y (u, v) in Euclidean space E_0, then

$$\varsigma = \frac{z - e_0\,(yz)}{\sqrt{1 + y^2}}$$

is the field of an infinitesimal deformation of the surface F:

$$x = \frac{y + e_0}{\sqrt{1 + y^2}}$$

in the elliptic space R. The field ς is trivial if, and only if, the field z is trivial.

<u>Proof of Theorem 1</u>. According to para. 1, for small λ the equations

$$x_1 = \rho\,(x + \lambda\varsigma), \quad x_2 = \rho\,(x - \lambda\varsigma)$$

give two isometric surfaces F_1 F_2 in elliptic space R. According to Theorem 1 in Chapter IV, the surfaces Φ_1 and Φ_2 in Euclidean space E_0 given by the equations

$$y_1 = \frac{x_1 - e_0(x_1 e_0)}{e_0\,(x_1 + x_2)}, \quad y_2 = \frac{x_2 - e_0(x_2 e_0)}{e_0\,(x_1 + x_2)},$$

are isometric. For small λ they are close together. But then, as pointed out in para. 1, the vector field

$$z = y_1 - y_2$$

is the field of an infinitesimal deformation of the surface Φ in Euclidean space, given by the equation

$$y = y_1 + y_2.$$

Clearly the field

$$z = \frac{1}{\lambda} (y_1 - y_2)$$

is also a field of infinitely small deformation of the surface Φ.

In the limit, when $\lambda = 0$, the surface Φ is given by the equation

$$y = \frac{x - e_0(xe_0)}{(e_0 x)},$$

and its field of infinitely small deformation is

$$z = \frac{\varsigma - e_0(\varsigma e_0)}{(e_0 x)}$$

According to Lemma 5 in Chapter III, if the field ς is trivial, i.e., is the velocity field of an infinitesimal motion, then z is also trivial.

Before concluding the proof of the theorem and showing the triviality of the field ς from that of the field z, we will prove Theorem 2.

<u>Proof of Theorem 2.</u> Since the field z is the field of an infinitesimal deformation of the surface Φ

$$y = y(u, v),$$

for small λ the equations

$$y_1 = \frac{1}{2} y + \lambda z, \quad y_2 = \frac{1}{2} y - \lambda z$$

give two isometric surfaces Φ_1 and Φ_2 in Euclidean space E_0. According to Theorem 2 in the foregoing section, the corresponding isometric surfaces in elliptic space R are F_1 and F_2 given by the equations:

$$x_1 = \rho\,(2y_1 + e_0\,(1 - \overset{2}{y_1} + \overset{2}{y_2})),$$
$$x_2 = \rho\,(2y_2 + e_0\,(1 - \overset{2}{y_2} + \overset{2}{y_1})).$$

If λ is small, the surfaces F_1 and F_2 are close together. Hence, according to the remark in para. 1, it follows that the surface F in elliptic space given by the equation

$$x = \rho\,(x_1 + x_2),$$

has the field

$$\varsigma = \rho\,(x_1 - x_2),$$

as the field of an infinitesimal deformation. At the same time,

$$\varsigma = \frac{1}{\lambda}\,\rho\,(x_1 - x_2).$$

will also be the field of an infinitesimal deformation.

Passing to the limit as $\lambda \to 0$, we obtain the surface F given by the equation

$$x = \frac{y + e_0}{\sqrt{1 + y^2}},$$

and its field of infinitely small deformation

$$\varsigma = \frac{z - e_0\,(yz)}{\sqrt{1 + y^2}}.$$

According to lemma 6 in Chapter III, if the field z is trivial, then the field ς is also trivial.

It can be seen from direct checking that comparison of the surfaces and their infinitesimal deformations established in Theorems 1 and 2 are reversible. It follows from this that field ς in Theorem 2 is trivial only if the field z is trivial, and vice-versa in theorem 1.

Both theorems are completely proved.

3. SOME THEOREMS ON INFINITESIMAL DEFORMATIONS OF SURFACES IN ELLIPTIC SPACE.

Theorem 1 enables us to extend many of the results relating to infinitely small deformations of surfaces in Euclidean space to surfaces in elliptic space.

We should point out that the correspondence between surfaces in elliptic and Euclidean space determined by Theorems 1 and 2 is not dependent on their infinitesimal deformations, but results from the geodesic mapping of one space on another (Chapter I).

In geodesic mapping of elliptic space on Euclidean (projective model of elliptic space) the convex surfaces in elliptic space become convex surfaces in Euclidean space.

Since closed convex surfaces in Euclidean space which do not contain plane parts, are infinitesimally rigid, that is to say do not admit any infinitesimal

deformations apart from trivial ones, closed convex surfaces in elliptic space which do not contain plane parts are infinitesimally rigid. We need only apply Theorem 1 to prove this statement.

Generally speaking, if a closed but not necessarily convex surface F in elliptic space has an infinitesimally rigid image when geodesically mapped into Euclidean space, it is rigid.

<u>Examples</u>. Let us use the term surface of type T in elliptic space for any surface F, whose convex part lies entirely on its convex shell. (The convex shell is the smallest convex surface containing F). The image of a surface of type T in Euclidean space under the geodesic mapping into elliptic space is a surface of type T. Aleksandrov has proved that analytic surfaces of type T in Euclidean space are rigid. According to Theorem 1, it follows from this that analytic surfaces of type T in elliptic space are also rigid.

It is known that if a convex surface in Euclidean space not containing pieces of a plane, has a single-valued projection onto a point 0, and the distances between the points of its edge and 0 are stationary, the surface is infinitesimally rigid. There is a corresponding theorem for elliptic space. The stationary nature of the distances between the edge points and the point 0 when changing to Euclidean space follows from Lemma 5 in Chapter III when applied to the line segments joining the points on the edge of the surface and 0. It follows from this, in particular, that convex caps are rigid within the class of caps in elliptic space.

I. N. Vekua has proved that a strictly convex surface in Euclidean space, is infinitesimally rigid,

provided the curvatures of the curves bounding it are stationary. This theorem can be extended to strictly convex surfaces in elliptic space. It also follows from Theorem 1. By way of proof we should point out that in infinitesimal deformations with stationary curvature of the curves bounding the surface, the osculating circles of these curves effect infinitesimal motions, and according to Lemma 5 in Chapter III, the curves in Euclidean space corresponding to these circles also effect infinitesimal motions. Since the order of the osculation is maintained in geodesic mapping, the curvature of the edge of the surface in Euclidean space will be stationary. The theorem on infinitesimal rigidity for surfaces in elliptic space thus follows from I. N. Vekua's theorem.

S. E. Cohn-Vossen has constructed examples of non-rigid closed surfaces in Euclidean space. Theorem 2 makes it possible to transfer these examples to elliptic space.

N. V. Efimov has proved the existence of surfaces rigid "locally", i.e., rigid in the neighborhood of a given point no matter how small. According to Theorem 2, it follows from this that there are such surfaces in elliptic space.

Sauer has proved that by knowing infinitesimal deformations of the surface F in Euclidean space, infinitesimal deformations of any surface obtained by projective transformation of F can be found explicitly. Since in the geodesic mapping of elliptic space to Euclidean space, projective transformations of figures correspond, Sauer's result can be transferred to surfaces in elliptic space.

All of the infinitesimal deformations of surfaces of the second degree are known in Euclidean space. Since in the geodesic mapping of Euclidean space into elliptic space surfaces of the second degree are transformed to surfaces of the second degree, Theorem 2 makes it possible to find all the infinitely small deformation surfaces of the second order in elliptic space.

We should point out in conclusion that Lemma 5 and 6 in Chapter III enable us to extend different theorems on rigidity of polyhedra in Euclidean space to polyhedra in elliptic space, for example, Cauchy's theorem of the rigidity of closed convex polyhedra.

CHAPTER VI

Single-Value Definiteness of General Convex Surfaces in Elliptic Space

Theorem 1 in Chapter IV, on the transformation of regular isometric surfaces in elliptic space can be extended to general convex surfaces. This makes it possible to obtain various theorems on uniqueness in elliptic space as consequences of the corresponding theorems for convex surfaces in Euclidean space.

1. A LEMMA ON RIB POINTS ON A CONVEX SURFACE

We will say that the plane α intersecting the convex surface F intersects its ribs if at each rib point on the surface F lying in the plane α, the rib of the tangent dihedral angle intersects the plane α (and does not lie in it)*. The following lemma is valid.

Lemma 1. Almost all planes intersecting the convex surface F in elliptic space intersect its ribs.

Proof. First, let us point out that the lemma need only be proved for convex surfaces in Euclidean

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*A point on a convex surface is called a regular point if its tangent cone is a plane, a rib point if its tangent cone consists of two half planes, and a conical point otherwise — Ed.

space. To see that this is so we need only examine the surface in Euclidean space corresponding to F in the geodesic mapping of elliptic space on Euclidean space.

Further, it is clear that the lemma need only be proved for small pieces of the surface. And, therefore, without loss of generality, it can be supposed that the surface F is uniquely projected onto the plane xy in the square K: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

Let us take small numbers σ and ω satisfying the condition $\omega <<< \sigma$, and divide the square K into smaller squares k by the straight lines $x = \frac{m'}{2^n}$, $y = \frac{m''}{2^n}$, m', m'' $< 2^n$. Let us designate the piece of the surface F projected onto the square k as F_k, and use K_ω to designate the totality of the squares k into which the conical points of the surface F with curvature greater than ω are projected. The part of the square K not covered by the squares K_ω, will be broken down again into smaller squares k', and we will use K_σ to designate the totality of squares k', inside which the rib points of the surface F with an external angle at the rib greater than σ are projected.

The directions of the ribs with the angle greater than σ on a piece F_k, hardly differ, provided ω and the squares k' are small. Indeed, if not, there would be conical points with curvature greater than ω on the part of the surface $F - F_{K_\omega}$, which is impossible.

It follows from this that the measure of the set of planes intersecting a piece F_k, of the surface, and

containing at least one rib with an angle greater than σ, is not greater than $c\varepsilon\delta$, where δ is the side of the square k', c is a constant which is solely a function of the largest angle of inclination of the planes of support F to the plane xy, and ε is small whenever ω and δ are small.

The value

$$\sum_{k' \subset K_\sigma} \delta_\sigma$$

is bounded by a constant M, which is solely a function of the integral of the mean curvature of the surface [2, Ch. VI]. It follows from this that the measure μ_σ of the set of planes intersecting the surface $F - F_{K_\omega}$ and containing at least one rib with an external angle greater than σ does not exceed

$$\frac{cM\varepsilon}{\sigma} .$$

And since ε is as small as required, $\mu_\sigma = 0$.

Changing to finer subdivisions in the squares k, we can conclude that the measure of the set of planes intersecting the surface F and containing a rib with an angle greater than σ is equal to zero.

Finally, letting $\sigma \to 0$, one sees that almost all planes intersecting the surface intersect its ribs and the lemma is proved.

2. TRANSFORMATION OF ISOMETRIC DIHEDRAL ANGLES AND CONES

We will use F_1 and F_2 to designate two isometric, similarly oriented convex surfaces in the region R_0 in elliptic space R, uniquely projected from the point e_0 (1, 0, 0, 0), and visible from within at this point*. We will use Φ_1 and Φ_2 to denote the corresponding surfaces in Euclidean space E_0 given by the equations

$$y_1 = \frac{x_1 - e_0(x_1 e_0)}{e_0 (x_1 + x_2)}, \quad y_2 = \frac{x_2 - e_0(x_2 e_0)}{e_0 (x_1 + x_2)}, \qquad (*)$$

where x_1 and x_2 are isometrically corresponding points on the surface F_1 and F_2. Since the surfaces F_1 and F_2 are projected uniquely from the point e_0, Φ_1 and Φ_2 are also projected uniquely from the origin of the coordinates. Indeed, any straight line in R passing through the point e_0 is given by the equation

$$x = \rho (e_0 + \lambda a) \qquad (\lambda - \text{parameter}),$$

and any straight line in E_0 passing the origin of the coordinates is given by equation

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*That is, every geodesic in R_0 beginning from e_0 intersects F_1 in at most one point and the surface "turns inward" toward 0. — Ed.

$$y = \mu b \quad (\mu - \text{parameter}).$$

It is easy to see that if the equation

$$x_1 = \rho (e_0 + \lambda a)$$

in λ does not have more than one solution for any a, the equation in μ

$$\frac{x_1 - e_0(x_1 e_0)}{e_0 (x_1 + x_2)} = \mu b$$

does not have more than one solution either for any b. And this means that the surface Φ_1 is uniquely projected from 0. The same conclusion can be drawn with regard to the surface Φ_2.

Let F_1 and F_2 be regular surfaces. Then, as found in Chapter IV, the surfaces Φ_1 and Φ_2 are locally convex. Let us show that each of these surfaces has its internal side turned towards the origin, i.e., that it is visible from inside at this point.

Let us use a_0 to designate a point on the surface F_1, a_1 and a_2 to denote unit tangent vectors at this point, and a_3 to denote the external normal. The corresponding vectors for the surfaces F_2 and Φ_1 will be designated b_0, b_1, b_2, b_3, and c_0, c_1, c_2, and c_3. We have (Chapter IV, para. 3):

$$c_0 = \frac{1}{\lambda_0} (a_0 - e_0 (a_0 e_0)), \ c_3 = \frac{1}{\lambda_0^4} (e_0, \ \lambda_0 a_1 - a_0 \lambda_1, \ a_2 \lambda_0 - a_0 \lambda_2).$$

Hence,

$$c_0 c_3 = \frac{1}{\lambda_0^3} (e_0 a_0 a_1 a_2) = -\frac{1}{\lambda_0^3} (a_3 e_0).$$

Since the surface F_1 is visible from inside at the point e_0, $e_0 a_3 < 0$, and, consequently $c_0 c_3 > 0$. This means that the surface Φ_1 is visible from inside at 0. Similarly it can be shown that the surface Φ_2 is visible from inside at 0.

Now let F_1 and F_2 be two isometric, identically oriented dihedral angles, and A_1 and A_2 be corresponding points on the ribs of the angles (the ribs of the angles need not correspond). Let us plot a line on the surface of the angle F_1 corresponding to the rib of the angle F_2, and a line corresponding to the rib of the angle F_1 on the surface of the angle F_2. Thus, each of the dihedral angles F_1, F_2 will be divided into four plane angles with vertices A_1 and A_2, respectively.

According to Formulae (*), the angles in Euclidean space corresponding to F_1 and F_2 are tetrahedral angles Φ_1 and Φ_2 (they may degenerate into dihedral ones, if the ribs of F_1 and F_2 are corresponding under the isometry). It is asserted that the tetrahedral angles Φ_1 and Φ_2 are convex and visible from inside the point O.

The proof is simple. The angles F_1 and F_2 can be approximated by regular cylindrical isometric surfaces $\widetilde{F}_1$ and $\widetilde{F}_2$. The corresponding convex surfaces $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ in Euclidean space are visible from inside point 0. When passing to the limit, where $\widetilde{F}_1 \rightarrow F_1$, and $\widetilde{F}_2 \rightarrow F_2$, the surfaces $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ should converge towards the convex surfaces visible from inside 0. But the tetrahedral angles Φ_1 and Φ_2 are the limits of $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$.

Let F_1 and F_2 be two isometric convex cones. It is asserted that the corresponding surfaces Φ_1 and Φ_2 in Euclidean space E_0 are convex cones, each of which is visible from inside the origin of the coordinates 0.

The fact that Φ_1 and Φ_2 are cones is clear, since the straight lines made isometrically corresponding by the formulae (*) are mapped into straight lines as was shown in Chapter III. We will approximate the cones F_1 and F_2 with the isometric cones $\widetilde{F}_1$ and $\widetilde{F}_2$ having regular surfaces. In Euclidean space they correspond to regular convex cones visible from inside point 0. A corresponding conclusion can be reached for the cones Φ_1 and Φ_2 by the limiting process $\widetilde{F}_1 \rightarrow F_1$ and $\widetilde{F}_2 \rightarrow F_2$.

Let us now consider the case of general convex surfaces F_1 and F_1 near the isometrically corresponding conical points A_1 and A_2. We will show

that the surfaces Φ_1 and Φ_2 at the corresponding points B_1 and B_2 are locally convex.

Let us plot the tangent cones B_1 and B_2 of the surfaces F_1 and F_2 at the points A_1 and A_2. The isometry of the surfaces induces an isometry of the cones. Let x_1 be a point on the surface F_1 close to A_1, and x_2 be an isometrically corresponding point on F_2. Let us join the points A_1 and x_1 by a segment and on its semi-tangent at the A_1 mark off the segment Δs equal to the distance between A_1 and x_1. Let x_1 be the end of this segment. As shown in para. 3, Chapter I,

$$x_1 = \bar{x}_1 + \varepsilon_1 \Delta s,$$

where $\varepsilon_1 \to 0$, when $x_1 \to A_1$. Making a corresponding plotting on the surface F_2 we obtain

$$x_2 = \bar{x}_2 + \varepsilon_2 \Delta s.$$

Substituting these expressions for x_1 and x_2 into Formulae (*), we obtain

$$y_1 = \frac{\bar{x}_1 - e_0\,(\bar{x}_1 e_0)}{e_0\,(\bar{x}_1 + \bar{x}_2)} + \bar{\varepsilon}_1 \Delta s, \quad y_2 = \frac{\bar{x}_2 - e_0\,(\bar{x}_2 e_0)}{e_0\,(\bar{x}_1 + \bar{x}_2)} + \bar{\varepsilon}_2 \Delta s.$$

Since the first terms on the right hand sides give the points of the cones corresponding to B_1 and B_2 and the second terms are of a higher order than Δs, these

cones will be tangent cones of the surfaces Φ_1 and Φ_2 at the points B_1 and B_2 corresponding to A_1 and A_2.

It follows from this that the surfaces Φ_1 and Φ_2 at points corresponding to conical points on the surfaces F_1 and F_2 are locally convex. The same conclusion can clearly be drawn in the case of the rib points A_1 and A_2, if the directions of the ribs do not correspond in isometry.

But if the ribs at the points A_1 and A_2 do correspond in the isometry, we can still claim the local convexity of a plane section of the surface Φ_1 at the point B_1, if the rib of the tangent dihedral angle at B_1 does not lie in the secant plane.

3. LOCAL CONVEXITY OF THE SURFACES Φ_1 AND Φ_2 AT SMOOTH POINTS

We will say that the surface Φ_i is weakly convex at the point B and has its inside turned toward the point 0 if we can draw a plane β through the point B in such a way that any sphere touching β at B and situated on the opposite side of the plane β from the point 0 is locally supported, i.e., a small neighborhood of the point B on the surface lies outside the sphere.

Let us show that the surface Φ_1 is weakly convex at the smooth point B_1 and has its inside turned towards

the point 0. For this we should point out first and foremost that the points A_1 and A_2, corresponding to B_1, on the surfaces F_1 and F_2 should be smooth.

For purposes of convenience in the forthcoming operations, we will designate the points A_1 and A_2 on the surfaces F_1 and F_2 by their vectors x_1 and x_2. On the surface F_1 we will take a point $x_1 + \Delta x_1$ close to x_1, and on the surface F_2 we will take an isometrically corresponding point $x_2 + \Delta x_2$. Let us join the points x_1 and $x_1 + \Delta x_1$ by the segment γ_1 on F_1, and the points x_2 and $x_2 + \Delta x_2$ by the corresponding segment γ_2 on F_2.

Taking into account the properties of segments on a convex surface described in para. 3 in Chapter I, we can put down the following expansions for the vectors $x_1 + \Delta x_1$ and $x_2 + \Delta x_2$:

$$x_1 + \Delta x_1 = \rho\left(\left(1 - \frac{\Delta s^2}{2!}\right) x_1 + \Delta s \tau_1 + \delta_1 \nu_1 + \varepsilon_1\right);$$

$$x_2 + \Delta x_2 = \rho\left(\left(1 - \frac{\Delta s^2}{2}\right) x_2 + \Delta s \tau_2 + \delta_2 \nu_2 + \varepsilon_2\right),$$

where τ_1 and τ_2 are unit tangent vectors of the geodesics at the points x_1 and x_2; ν_1 and ν_2 are internal unit normals of the surfaces F_1 and F_2 at these points; δ_1 and δ_2 are the distances between the points $x_1 + \Delta x_1$ and $x_2 + \Delta x_2$ and the tangent planes

of the surfaces F_1 and F_2 at the points x_1 and x_2, and ϵ_1/δ_1 and ϵ_2/δ_2 tend to zero together with Δs — the lengths of the segments γ_1 and γ_2.

Substituting these expressions into the formulae (*), we obtain

$$\Delta y_1 = \frac{\Delta s}{e_0\,(x_1 + x_2)}\{-y_1\,(\bar{e}_0\tau_1 + \bar{e}_0\tau_2) + \tau_1 - \bar{e}_0\,(\bar{e}_0\tau_1)\} +$$

$$+ \frac{\Delta s^2 e_0\,(\tau_1 + \tau_2)}{e_0\,(x_1 + x_2)}\{y_1\,(e_0\tau_1 + e_0\tau_2) - \tau_1 + e_0\,(e_0\tau_1),\ +$$

$$+ \frac{1}{e_0\,(x_1 + x_2)}\{-(\delta_1 e_0\nu_1 + \delta_2 e_0\nu_2)\,y_1 + \delta_1\nu_1 - \delta_1 e_0\,(e_0\nu_1)\} +$$

$$+ \varepsilon'\delta_1 + \varepsilon''\delta_2 + \varepsilon\Delta s^2,$$

where ϵ', ϵ'' and ϵ tend to zero with Δs.

In order to make it clear that the surface Φ_1 is weakly convex at the point B_1, we only need to show that $\Delta y_1 n$, where n is the normal to Φ_1, at the point B_1, keeps its sign with an accuracy to within the order of $\epsilon \Delta s^2$.

By multiplying Δy_1 by the normal n of the surface Φ_1 at the point B_1 we should obtain a quantity of a higher order of smallness than Δs. Hence, taking into account that

$$\delta_1/\Delta s \to 0 \ \text{ and } \ \delta_2/\Delta s \to 0 \ \text{ as } \ \Delta s \to 0, \ \text{we conclude}$$

$$n\,\{-y_1\,(e_0\tau_1 + e_0\tau_2) + \tau_1 - e_0\,(e_0\tau_1) = 0.$$

And for $n\Delta y_1$ we obtain

$$n \Delta y_1 = \frac{\delta_1}{e_0 (x_1 + x_2)} \left(-y_1 (e_0 \nu_1) + \nu_1 \right) n +$$

$$- \frac{\delta_2}{e_0 (x_1 + x_2)} (e_0 \nu_2) (y_1 n) + \varepsilon \Delta s^2.$$

As established in para. 3 of Chapter IV, the expressions

$$\left(-y_1 (e_0 \nu_1) + \nu_1 \right) n \text{ and} - (e_0 \nu_2) (y_1 n)$$

differ from zero and have the same signs. And we can conclude that the surface Φ_1 is weakly convex at the point B_1.

That Φ_1 has its inside turned towards 0 at the point B_1 follows from the fact that this happens in the case of regular surfaces F_1 and F_2. Indeed, let us take two isometric regular convex surfaces $\widetilde{F}_1$ and $\widetilde{F}_2$ osculating with F_1 and F_2 at the points A_1 and A_2, with their internal sides turned towards e_0. The surface $\widetilde{\Phi}_1$ will then be convex and turned with the inside towards 0. And since the expressions

$$\left(-y_1 (e_0 \nu_1) + \nu_1 \right) n \text{ and} - (e_0 \nu_2) (y_1 n)$$

in the case of the surfaces $\widetilde{F}_1$ and $\widetilde{F}_2$ are the same as for F_1 and F_2, the surface Φ_1, like $\widetilde{\Phi}_1$, has its internal side turned towards 0 at point B_1.

The weak convexity of the surface Φ_2 at a smooth point is established in a similar fashion.

4. CONVEXITY AND ISOMETRY OF THE SURFACES Φ_1 AND Φ_2

Let us map the region R_0 in elliptic space onto Euclidean space E_0 by mapping the point x with the Weierstrass coordinates x_0, x_1, x_2 and x_3 to the point with Cartesian coordinates x_1/x_0, x_2/x_0 and x_3/x_0. In Euclidean space E_0 the surfaces F_1 and F_2 are represented by convex surfaces, uniquely projected from the origin of the coordinates and with their internal sides towards it. The correspondence between the points on the surfaces F_1 and Φ_1, F_2 and Φ_2, determined by the formulae (*), is achieved here by simple projection from the origin of the coordinates.

We assert that a piece Φ' of the surface Φ_1 (Φ_2) cut off by a convex cone V with its apex at the origin is convex. So as not to have to introduce new symbols, we will assume that $\Phi' \equiv \Phi_1$. Clearly the convexity of the surface Φ_1 will be proved if it is established that any plane α spanned by two generators of the cone V intersects the surface in a convex curve with its convexity towards the apex of the cone.

Let us assume that the plane α intersects the ribs of the surface F_1 (para. 1). Then at all points along the line γ of intersection of the plane and the surface Φ_1 this surface is at least weakly convex (para. 2 and 3).

Let us use $\overline{\gamma}$ to designate the convex shell of the curve γ. The points of contact with the generators of the cone V lying in the plane α divide $\overline{\gamma}$ into two parts. The part which is further from the point 0 will be termed γ'. We are required to proved that γ coincides with γ'.

Unless γ coincides with γ', the part of γ' which does not belong to γ consists of rectilinear segments with ends on the curve γ. Let g be one of these segments and let $\widetilde{\gamma}'$ be the corresponding segment on the curve γ under central projection from 0.

Let us take points P and Q on the continuation of the segment g on either side of it and draw an arc of the circumference through them with a radius large enough for the region bounded by it and the sector PQ to contain the point 0. Now let us imagine that the circumference is deformed in such a way that its center moves in a direction perpendicular to the segment g, and that during the deformation it passes through the points P and Q. Then, clearly, there will be a moment when the circumference of the internal side will touch the curve $\widetilde{\gamma}'$. But this is impossible by virtue of the weak convexity of the surface Φ along the curve γ. We are faced with a contradiction, hence the curve γ is convex.

We assumed that the plane α intersected the ribs of the surface F_1. We will now abandon this limitation. Let α be an arbitrary plane. According to the lemma in para. 1, an arbitrarily small shift, of the surface F_1 can be used to make α intersect the ribs of the surface F_1. And since Φ_1 is a continuous

function of the position of F_1, any plane α (spanned by two generators of V) intersects Φ_1 along a convex curve. Thus, the convexity of the surface Φ_1 is proved.

The convexity of the surface Φ_2 is proved in a similar way.

Now let us prove that the surfaces Φ_1 and Φ_2 are isometric. Let us take an arbitrary, rectifiable curve $\overline{\gamma}_1$ on the surface Φ_1. Since the total angle of the cone projecting it from the point 0 is finite, there is a corresponding rectifiable curve γ_1 on F_1. (The correspondence of the points on Φ_1 with their pre-image in F_1 (regarded to be E_0) is brought about by projection from the point 0).

Let the rectifiable curve γ_2 on the surface F_2 correspond isometrically to the curve γ_1 on F_1. It follows from this that the curve $\overline{\gamma}_2$ on the surface Φ_2 corresponding to $\overline{\gamma}_1$ is rectifiable. Let us show that the curves $\overline{\gamma}_1$ and $\overline{\gamma}_2$ have identical lengths. This will establish the isometry of the surfaces Φ_1 and Φ_2.

Since the curve γ_1 on F_1 is rectifiable, the vector function x_1 (s) where x is the arc along the curve γ_1, satisfies a Lipschitz condition. For the same reason the vector function x_2 (s) satisfies a Lipschitz

condition. It follows from this that the vector functions y_1 (s) and y_2 (s) determined by Formulae (*) also satisfy this condition and the lengths of the curves $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are found by means of the integrals

$$\int \left|\frac{dy_1}{ds}\right| ds, \quad \int \left|\frac{dy_2}{ds}\right| ds.$$

In order to show that the lengths of the curves $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are identical, we need only show that $|y_1'(s)| = |y_2'(s)|$ for almost all s.

Let the derivatives x_1' (s) and x_2' (s) exist at the corresponding points x_1 (s) and x_2 (s) on the curves γ_1 and γ_2. We will show that the derivatives y_1' (s), y_2' (s) and $|y_1'(s)| = |y_2'(s)|$ then exist.

The existence of the derivatives x_1' (s) and x_2' (s) indicates that the curves γ_1 and γ_2 have tangents at the points x_1 and x_2. Let us map respectively the curves γ_1 and γ_2 on the tangents t_1 and t_2 at the points x_1 and x_2, comparing the points at the same distances from x_1 and x_2. Thus, the isometric correspondence of the curves γ_1 and γ_2 gives rise to the isometric correspondence between the tangents t_1 and t_2.

Let $\overline{x}_1$ (s) and $\overline{x}_2$ (s) be the points on the tangents corresponding at a distance s from x_1 and x_2, respectively. The equations

$$\bar{y}_1 = \frac{\bar{x}_1 - e_0\,(\bar{x}_1 e_0)}{e_0\,(\bar{x}_1 + \bar{x}_2)}, \quad \bar{y}_2 = \frac{\bar{x}_2 - e_0\,(\bar{x}_2 e_0)}{e_0\,(\bar{x}_1 + \bar{x}_2)}$$

give the two lines (Chapter III) in Euclidean space E_0, and the correspondence between them, which compares equal values of the parameter is isometric. It therefore follows that:

$$|\bar{y}_1'\,(s)| = |\bar{y}_2'\,(s)|.$$

Taking it into account that $x_1\,(s) = \bar{x}_1(s)$, $x_2\,(s) = \bar{x}_2\,(s)$, $x_1'\,(s) = \bar{x}_1'\,(s)$, $x_2'\,(s) = \bar{x}_2'\,(s)$, and comparing the derivatives $y_1'\,(s)$ and $\bar{y}_1'\,(s)$, $y_2'\,(s)$ and $\bar{y}_2'\,(s)$, expressed in terms of x_1, x_1', $\bar{x}_1'$, x_2, x_2', $\bar{x}_2'$, we obtain $y_1'\,(s) = \bar{y}_1'\,(s)$, $y_2'\,(s) = \bar{y}_2'(s)$. Consequently,

$$|y'\,(s)| = |y_2'\,(s)|$$

almost everywhere, and the curves $\overline{\gamma}_1$ and $\overline{\gamma}_2$ have the same lengths.

Since the curve $\overline{\gamma}_1$ was taken arbitrarily, the surfaces Φ_1 and Φ_2 are isometric.

5. VARIOUS THEOREMS ON UNIQUENESS OF CONVEX SURFACES IN ELLIPTIC SPACE

Theorem 1. Closed isometric convex surfaces in elliptic space are congruent.

<u>Proof</u>. Let F_1 and F_2 be two closed isometric convex surfaces in elliptic space R. Without loss of

generality, it can be supposed that these surfaces are identically oriented, lie in the region R_0 of space and the point e_0 lies inside both surfaces.

This arrangement of surfaces may be obtained in the following way. If the surfaces are oppositely oriented, let one of them be reflected in a plane. After this transfer both surfaces by motions to the region R_0 of elliptic space.

If in the process one of the surfaces, for example F_1, no longer contains the point e_0, we can join it to e_0 by a shortest segment g in R_0. Let A_1 be the end of this segment belonging to the surface. In R_0 we will draw two planes of support of the surface perpendicular to the geodesic containing g. They intersect along a line h belonging to the boundary of R_0. Let the surface F_1 rotate about this line until the point A_1 coincides with e_0. In this process, the surface clearly remains within R_0. An arbitrarily small motion can transfer the surface from this position to a position such that the point e_0 is inside it.

As established in para. 4, a convex cone V with its apex at the point 0 cuts out a convex surface from the surfaces Φ_1 and Φ_2. It follows from this that the surfaces Φ_1 and Φ_2 are closed and convex.

It was also established in para. 4 that the surfaces

Φ_1 and Φ_2 are isometric. According to the theorem on uniqueness for general convex surfaces [2], Φ_1 and Φ_2 are congruent. And it follows from this, according to the lemma 4 in Chapter III that F_1 and F_2 are congruent.

The theorem is proved.

Theorem 2. Convex isometric caps in elliptic space are congruent.

Indeed, if F_1 and F_2 are two isometric convex caps, then, by supplementing each of them with its reflection in the bounding plane, we obtain two closed isometric convex surfaces. According to Theorem 1, the surfaces are congruent. Consequently, the caps are equal as congruent.

Theorem 3. Let us assume we have two isometric convex surfaces F_1 and F_2 in the region R_0 of elliptic space, each of which is visible from inside at (outside) e_0, and that the point e_0 is outside the convex shells of the surfaces.

Then, if the isometrically corresponding points on the boundries of the surfaces F_1 and F_2 are the same distance away from e_0, the surfaces F_1 and F_2 are congruent.

This theorem, like Theorem 1, can be reduced to the corresponding theorem for surfaces in Euclidean

space. We will consider the case in which the surfaces F_1 and F_2 are visible from inside at e_0. To consider a case in which they were visible from the outside we would require a certain modification of the auxilliary considerations given in paras. 2 and 3.

The surfaces Φ_1 and Φ_2 in Euclidean space corresponding to F_1 and F_2 are isometric, convex within a small neighborhood of each point, and have their outer sides facing point 0. Let us show that the isometrically corresponding points on the boundaries of the surfaces Φ_1 and Φ_2 are the same distance from 0. The isometry of the surfaces F_1 and F_2 can naturally be extended to an isometry of rectilinear segments joining the points e_0 with the corresponding points on the boundaries. According to formula (*), the rectilinear segments joining e_0 and the boundary of F_1 and F_2 correspond to equal rectilinear segments joining the origin of the coordinates 0 to the corresponding points on the boundaries of the surfaces Φ_1 and Φ_2 (Chapter III).

The point 0 lies outside the convex shells of the surfaces Φ_1 and Φ_2, because the formulae (*), when applied to line segments reduces to the projection of the latter on E_0 and its extension (i.e., its ideal points — Ed.).

According to the theorem established by A. D. Aleksandrov and E.P. Sen'kin [5], the surfaces Φ_1

and Φ_2 are congruent. The congruence of F_1 and F_2 follows from this. Admittedly, in their publication the congruence of the surfaces is established on the assumption of their regularity, and also for a case in which each of them is a polyhedron. But the result is valid for general convex surfaces, which can be seen from the following argument.

The cited publication shows without any other assumptions, apart from convexity, that the surfaces Φ_1 and Φ_2, unless they are congruent, may be arranged in such a way that

$$d_1 < d_2,$$

at the edge of them, and at certain internal points

$$d_1 > d_2,$$

where d_1 and d_2 are the distances of the isometrically corresponding points of the surfaces from 0.

Let us use P_1 and P_2 to designate the isometrically corresponding triangulation of the surfaces Φ_1 and Φ_2 which are now supposed to be such that $d_1 < d_2$ along their boundaries, and such that there are points inside at which $d_1 > d_2$. Let us extend each of the surfaces Φ_1 and Φ_2 until they are closed and designate the surfaces obtained as Φ_1 and Φ_2.*

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*By adding the rays which connect the boundaries to the origin. — Ed.

We will subject each of them to a fine triangulation, but in such a way that the triangulations coincide on P_1 and P_2, i.e., in such a way that they correspond isometrically.

Each of the geodesic triangles of the surfaces Φ_1 and Φ_2 can be replaced by a flat triangle with the same sides. And from these triangles we will form the closed convex polyhedra Φ_1 and Φ_2. This can be done by A. D. Aleksandrov's theorem.

It follows from the theorem on uniqueness of closed convex surfaces that as the triangulations of the surfaces Φ_1 and Φ_2 become finer, the polyhedra Φ_1 and Φ_2 converge toward the surfaces. Hence, when the triangulations are fairly fine, the parts of the polyhedra Φ_1 and Φ_2 corresponding to the surfaces Φ_1 and Φ_2 also tolerate an arrangement by which

$$d_1 < d_2,$$

at their boundaries, and for several internal points

$$d_1 > d_2.$$

But as was shown in [5], this is impossible for polyhedra. We have come up against a contradiction and the theorem is valid without any further assumptions about the surfaces Φ_1 and Φ_2, apart from convexity.

Let us consider another theorem.

Theorem 4. If the convex isometric surfaces F_1 and F_2 in elliptic space are obtained from closed surfaces $\bar{F}_1$ and $\bar{F}_2$ by cutting out regions isometric to a subset of a plane, which are bounded by the curves γ_1 and γ_2 of bounded swerve, and if the curvatures of F_1 and F_2 on these curves are equal to zero, the surfaces F_1 and F_2 are congruent.

This theorm follows from Theorem 1, since the isometric correspondence of the surfaces F_1 and F_2 can be extended to an isometric correspondence of closed surfaces $\bar{F}_1$ and $\bar{F}_2$.

Indeed, the curves γ_1 and γ_2 have identical swerve away from the regions F_1 and F_2 on the surfaces $\bar{F}_1$ and $\bar{F}_2$ on corresponding arcs. Since the curvatures of $\bar{F}_1$ and $\bar{F}_2$ on the curves γ_1 and γ_2 are equal to zero, the curves γ_1 and γ_2 have equal swerves away from the regions bounded by them. It follows from this that these regions are isometric and the isometric correspondence of $\bar{F}_1$ and $\bar{F}_2$ in these regions is induced by the isometric correspondence of the curves bounding them.

CHAPTER VII

Regularity of Convex Surfaces with a Regular Metric

This section will deal with the following problem. To what extent does the regularity of the intrinsic metric of a convex surface, i.e., the regularity of the functions E(u, v), F(u, v) and G(u, v) of its line element imply the regularity of the surface itself, i.e., the regularity of $x_0(u, v)$, $x_1(u, v)$, $x_2(u, v)$, $x_3(u, v)$.

1. THE DEFORMATION EQUATION FOR SURFACES IN ELLIPTIC SPACE.

Let F be a regular surface in elliptic space and ζ be one of the Weierstrass coordinates for an arbitrary point on the surface. Let us devise an equation which is satisfied by ζ, provided the line element of the surface is

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2.$$

For this let us turn to the derivation formulae. We have:

$$x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v - Ex + L\xi.$$
$$x_{uv} = \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v - Fx + M\xi,$$
$$x_{vv} = \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v - Gx + N\xi.$$

Multiplying these formulae by the unit coordinate

vector e of the ζ axis, we obtain

$$\zeta_{uu} = \Gamma_{11}^1 \zeta_u + \Gamma_{11}^2 \zeta_v - E\zeta + L\,(\xi e),$$

$$\zeta_{uv} = \Gamma_{12}^1 \zeta_u + \Gamma_{12}^2 \zeta_v - F\zeta + M\,(\xi e), \qquad (*)$$

$$\zeta_{vv} = \Gamma_{22}^1 \zeta_u + \Gamma_{22}^2 \zeta_v - G\zeta + N\,(\xi e).$$

Let us transfer all the terms in the right-hand sides of these equalities, except for the final ones, to the left-hand side. Then multiply the first and third equalities, square the second, and subtract. This gives us

$$(\zeta_{uu} - \Gamma_{11}^1 \zeta_u - \Gamma_{11}^2 \zeta_v + E\zeta)(\zeta_{vv} - \Gamma_{22}^1 \zeta_u - \Gamma_{22}^2 \zeta_v + G\zeta) -$$
$$- (\zeta_{uv} - \Gamma_{12}^1 \zeta_u - \Gamma_{12}^2 \zeta_v + F\zeta)^2 = (LN - M^2)\,(\xi e)^2.$$

Let us express $(\xi e)^2$ in terms of ζ. We have:

$$\xi = \frac{(x x_u x_v)}{|(x x_u x_v)|}, \qquad (\xi e)^2 = \frac{(e x x_u x_v)^2}{(x x_u x_v)^2}.$$

Applying the vector identities in Chapter I, para. 1 to the numerator and denominator of the expression $(\xi e)^2$, we obtain:

$$(x x_u x_v)^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & E & F \\ 0 & F & G \end{vmatrix} = EG - F^2,$$

$$(e x x_u x_v)^2 = \begin{vmatrix} 1 & \zeta & \zeta_u & \zeta_v \\ \zeta & 1 & 0 & 0 \\ \zeta_u & 0 & E & F \\ \zeta_v & 0 & F & G \end{vmatrix} = (1 - \zeta^2)(FG - F^2) - G\zeta_u^2 -$$
$$- E\zeta_v^2 + 2F\zeta_u \zeta_v.$$

Now observing that:

$$\frac{LN - M^2}{EG - F^2} = K - 1,$$

where K is the Gaussian curvature of the surface, we obtain the equation for ζ in the following form:

$$(\zeta_{uu} - \Gamma^1_{11}\zeta_u - \Gamma^2_{11}\zeta_v + E\zeta)(\zeta_{vv} - \Gamma^1_{22}\zeta_u - \Gamma^2_{22}\zeta_v + G\zeta) - (\zeta_{uv} - \Gamma^1_{12}\zeta_u - \Gamma^2_{12}\zeta_v + F\zeta)^2 = (K-1)\{(1-\zeta^2)(EG - F^2) - G\zeta_u^2 - E\zeta_v^2 + 2F\zeta_u\zeta_v\}.$$

The coefficients of this equation are expressed in terms of the coefficients of the line element E, F and G and their derivatives.

Instead of ζ, we will introduce into this equation a new function z associated with ζ by the equality

$$\zeta = \sin z.$$

It is easy to see that z has simple geometrical significance. Up to the sign, z is the distance from the point on the surface and to the plane $\zeta = 0$. Observing that

$$z_u = \frac{\zeta_u}{\sqrt{1 - \zeta^2}}, \qquad z_v = \frac{\zeta_v}{\sqrt{1 - \zeta^2}}, \qquad \operatorname{tg} z = \frac{\zeta}{\sqrt{1 - \zeta^2}};$$

$$\frac{\zeta_{uu}}{\sqrt{1 - \zeta^2}} = z_{uu} - z_u^2 \operatorname{tg} z;$$

$$\frac{\zeta_{uv}}{\sqrt{1 - \zeta^2}} = z_{uv} - z_u z_v \operatorname{tg} z;$$

$$\frac{\zeta_{vv}}{\sqrt{1 - \zeta^2}} = z_{vv} - z_v^2 \operatorname{tg} z,$$

we obtain an equation for z in the following form:

$$\{z_{uu} - \Gamma^1_{11}z_u - \Gamma^2_{11}z_v + \operatorname{tg} z\,(E - z^2_u)\}\,\{z_{vv} - \Gamma^1_{22}z_u - \Gamma^2_{22}z_v +$$
$$+ \operatorname{tg} z\,(G - z^2_v)\} - \{z_{uv} - \Gamma^1_{12}z_u - \Gamma^2_{12}z_v + \operatorname{tg} z\,(F - z_u z_v)\}^2 =$$
$$= (K - 1)\,\{(E - z^2_u)\,(G - z^2_v) - (F - z_u z_v)^2\}.$$

Let us note that this equation only differs from the one for deformation of the surfaces in Euclidean space in the last terms in braces on the left-hand side of the equality and by x - 1 in the expression (K - 1) on the right-hand side.

If z is substituted for ζ in Formulae (*), we obtain the following expressions for the coefficients L, M and N of the second fundamental form of the surface

$$L = \frac{1}{Q}\,\{z_{uu} - \Gamma^1_{11}z_u - \Gamma^2_{11}z_v + \operatorname{tg} z\,(E - z^2_u)\}\,;$$

$$M = \frac{1}{Q}\,\{z_{uv} - \Gamma^1_{12}z_u - \Gamma^2_{12}z_v + \operatorname{tg} z\,(F - z_u z_v)\,\}\,;$$

$$N = \frac{1}{Q}\,\{z_{vv} - \Gamma^1_{22}z_u - \Gamma^2_{22}z_v + \operatorname{tg} z\,(G - z^2_v)\},$$

where

$$Q^2 = \frac{(E - z^2_u)\,(G - z^2_v) - (F - z_u z_v)^2}{EG - F^2}\,.$$

Now let the parametrization u and v of the surface be geodesic polar, and consequently, the line element of the surface will be

$$ds^2 = du^2 + c^2 dv^2.$$

Then the Christoffel symbols take the following form:

$$\Gamma_{11}^1 = 0, \qquad \Gamma_{11}^2 = 0;$$

$$\Gamma_{12}^1 = 0, \qquad \Gamma_{12}^2 = \frac{c_u}{c};$$

$$\Gamma_{22}^1 = -cc_u, \qquad \Gamma_{22}^2 = \frac{c_v}{c}.$$

The equation for deformation takes the form

$$(r + \operatorname{tg} z\,(1 - p^2))\,(t + cc_u p - \frac{c_v}{c}\,q + \operatorname{tg} z\,(c^2 - q^2)) -$$

$$- (s - \frac{c_u}{c}\,q - \operatorname{tg} zpq)^2 = (K - 1)\,(c^2 - c^2 p^2 - q^2).$$

and the coefficients L, M and N are expressed by the formulae

$$L = c\frac{r + \operatorname{tg} z\,(1 - p^2)}{\sqrt{c^2 - c^2 p^2 - q^2}};$$

$$M = c\frac{s - \frac{c_u}{c}\,q - \operatorname{tg} zpq}{\sqrt{c^2 - c^2 p^2 - q^2}};$$

$$N = c\frac{t + cc_u p - \frac{c_v}{c}\,q + \operatorname{tg} z\,(c^2 - q^2)}{\sqrt{c^2 - c^2 p^2 - q^2}},$$

where r, s, t, p and q are Monge designations for derivatives of the function z.

2. EVALUATION OF THE NORMAL CURVATURES OF A REGULAR CONVEX CAP IN ELLIPTIC SPACE.

Lemma 1. Let F a regular* convex cap in elliptic space. Let the Gaussian curvature of the cap be everywhere greater than the curvature of space.

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* Here "regular" means of class C^4 at least. — Ed.

It is then possible to estimate the normal curvature of the cap at point x in terms of the intrinsic metric of the cap and the distance h (x) between the point x and the plane of the base of the cap only.

<u>Proof.</u> Let x be an arbitrary point on the cap and let γ be an arbitrary geodesic originating from x. Let us use $\varkappa_\gamma$ (x) to designate the normal curvature of the cap at the point x in the direction γ and let us consider the function

$$w_\gamma (x) = h (x) \varkappa_\gamma (x).$$

The function ω_γ (x) is non-negative and becomes zero at the boundary of the cap. W_γ (x) attains a maximum at an internal point in the cap x_0 for a geodesic γ_0 originating from x_0. Let us designate this maximum as ω_0.

Let us draw a geodesic $\overline{\gamma}_0$ perpendicular to γ_0 through the point x_0 and introduce the geodesic coordinate network uv in the neighborhood of the point x_0, taking the geodesics perpendicular to $\overline{\gamma}_0$ as the lines u = constant. As the parameters u and v let us take the arc lengths of the geodesics γ_0 and $\overline{\gamma}_0$, measured from the point x_0.

Let us use γ (x) to designate a geodesic of the family u = const. passing through the point x close to x_0, and introduce the function

$$w(x) = w_{\gamma (x)} (x)$$

This function also clearly attains a maximum at x_0 and this maximum is ω_0.

Let us derive the equation for deformation for F (para. 1) by taking the tangent plane of the cap at the point x_0 as the plane $z = 0$. For the sake of definiteness we will suppose that the cap is in the half space $z < 0$. Assuming that

$$\alpha = (1 - p^2)\operatorname{tg} z, \quad \beta = -\frac{c_u}{c}q - pq\operatorname{tg} z,$$

$$\gamma = cc_u p - \frac{c_v}{c}q + (c^2 - q^2)\operatorname{tg} z, \quad \delta = \left(\frac{c_{uu}}{c} + 1\right)(c^2 - c^2 p^2 - q^2),$$

we can write the equation for deformation in the form

$$(r + \alpha)(t + \gamma) - (s + \beta)^2 + \delta = 0.$$

Let us express the normal curvature $\varkappa_\gamma(x)$ in terms of z. To do this we can make use of the formula in para. 1 for the coefficient L of the second quadratic form. We obtain

$$\varkappa_\gamma(x) = c\frac{r + (1 - p^2)\operatorname{tg} z}{\sqrt{c^2 - c^2 p^2 - q^2}}.$$

From here

$$w = hc\frac{r + (1 - p^2)\operatorname{tg} z}{\sqrt{c^2 - c^2 p^2 - q^2}}.$$

Solving this equation for r, we obtain

$$r = \frac{w}{h}\sqrt{1 - p^2 - \frac{q^2}{c^2}} - (1 - p^2)\operatorname{tg} z.$$

On the basis of this expression for r, we can find its first and second derivatives with respect to u and v at the point x_0. According to our selection of the coordinate system $c = 1$, $c_u = c_v = c_{uv} = c_{vv} = 0$, $c_{uu} =$ —K, where K is the Gaussian curvature of the surface. The derivatives p and q are equal to zero, since the plane z = 0 touches the surface. The derivative s = 0, since M = 0 (the directions of the coordinate lines at the point x_0 are the principal directions). Finally, $\omega_u = \omega_v = 0$, since ω attains a maximum at the point x_0.

We have

$$r_u = \left(\frac{1}{h}\right)_u w, \quad r_v = \left(\frac{1}{h}\right)_v w;$$

$$r_{uu} = \frac{w_{uu}}{h} + w\left(\left(\frac{1}{h}\right)_{uu} - \frac{r^2}{h}\right) - r;$$

$$r_{vv} = \frac{w_{vv}}{h} + w\left(\left(\frac{1}{h}\right)_{vv} - \frac{t^2}{h}\right) - t.$$

Differentiating the equation for deformation at the point x_0 with respect to u, we obtain

$$rt_u + tr_u - K_u = 0$$

From here

$$t_u = \frac{K_u - tr_u}{r}.$$

Differentiating the equation for deformation with respect to u twice, we obtain

$$r_{uu}t + t_{uu}r + 2r_u t_u - 2s_u^2 - r^2 + 2K^2 - K - 1 - K_{uu} = 0.$$

Introducing the values of $r_u, r_v, t_u, s_u = r_v, r_{uu}$ and $t_{uu} = r_{vv}$ obtained above into this equality. Furthermore, let us replace ω by its value at the point x_0, rh. We then obtain:

$$\frac{1}{h}(w_{uu}\, t + w_{vv}\, r) - h\,(K - 1)\,(r^2 + t^2) - 2r^2 h^2 \left(\left(\frac{1}{h}\right)_u^2 + \left(\frac{1}{h}\right)_v^2\right) +$$
$$+ h\,(K-1)\left(\left(\frac{1}{h}\right)_{uu} + \left(\frac{1}{h}\right)_{vv}\right) + 2\left(\frac{1}{h}\right)_u hK_u - r^2 + 2K^2 - 3K +$$
$$+ 1 - K_{uu} = 0. \qquad (*)$$

Let us evaluate the second derivatives h_{uu} and h_{vv}. To do this we will take the plane of the base of the cap to be the plane $z = 0$ in the formulae for L and N (para. 1). The principal curvatures at the point x_0, which are equal to r and t, are then expressed in terms of h by the formulae:

$$r = -\frac{h_{uu} + \operatorname{tg} h\,(1 - h_u^2)}{\sqrt{1 - h_u^2 - h_v^2}},$$
$$t = -\frac{h_{vv} + \operatorname{tg} h\,(1 - h_v^2)}{\sqrt{1 - h_u^2 - h_v^2}}.$$

This gives us expressions for h_{uu} and h_{vv} which we will substitute into the equality (*).

Taking it into account that the first two terms in the left-hand side of the equality (*) are not positive, we obtain the inequality:

$$-r^2h^2 + Arh + B \geqslant 0,$$

where A and B are known bounded expressions containing the Gaussian curvature K and its derivatives, h and its first derivatives h_u and h_v.

Since rh = ω, it works out that

$$-w_0^2 + Aw + B \geqslant 0.$$

From this it follows that an upper bound may be established for $\overline{w}_0$. If this bound is designated $\overline{w}_0$, the normal curvature of the cap at a point at a distance h from its base will not exceed w_0/h. And the theorem in proved.

3. CONVEX SURFACES OF BOUNDED SPECIFIC CURVATURE IN ELLIPTIC SPACE.

Let F be a convex surface in elliptic space. We will define the specific extrinsic curvature of the set M on the surface F to be the ratio $\omega(M)/s(M)$, where ω (M) is the extrinsic curvature of the surface at the set M, and s (M) is the surface area of this set. We will say that the specific curvature of the surface is bounded from below (from above) when there is a positive constant c which is such that for any Borel set M on the surface ω (M)/s(M) $\leqslant c$ (respectively, ω (M)/s(M) $\geqslant c$).

A. D. Aleksandrov has proved the following two theorems for surfaces of bounded specific curvature in Euclidean space:

1) convex surfaces of specific curvature bounded from above do not contain either rib or conical points;

2) convex surfaces of specific curvature bounded from below are strictly convex, i.e., they cannot contain any rectilinear segments.

We would like to extend these theorems to cover the case of convex surfaces of bounded specific curvature in elliptic space.

Let us use R_ρ to designate a sphere of radius $\rho < \frac{\pi}{2}$ with its center at $e_0 = (1, 0, 0, 0)$ in elliptic space R. Let F be any convex surface contained inside the sphere R_ρ.

Let us map the elliptic space R geodesically in Euclidean space $x_0 = 1$, by mapping the point x in elliptic space to the point x/xe_0 in Euclidean space. If elliptic space is consected to be the three dimensional sphere, this mapping consists in projecting the sphere from its center to the tangent hyperplane at the point e_0. In this mapping, the convex surface F in R will correspond to the convex surface $\overline{F}$ in Euclidean space.

Let us designate the extrinsic curvature and area of the set M on the surface F by ω (M) and s (M), and the curvature and area of the corresponding set $\overline{M}$ on the surface $\overline{F}$ by $\overline{\omega}(\overline{M})$ and $\overline{s}$ $(\overline{M})$. The following lemma is then valid:

Lemma 2. There are positive constants A_1, B_1 and A_2, B_2, which are solely a function of ρ and are such that for any open set M on the surface F the inequalities

$$A_1 \leqslant \frac{s\,(M)}{\overline{s}\,(\overline{M})} \leqslant B_1, \quad A_2 \leqslant \frac{\omega\,(M)}{\overline{\omega}\,(\overline{M})} \leqslant B_2.$$

Proof. Without loss of generality, it can be supposed that the surface F is closed, since any convex surface can be extended until it is closed. Let us take a fairly dense net of points on the surface F and construct their convex shell. The shell is a polyhedron P. Let us establish the correspondence of the points of P and of F by projecting from a point 0 inside the surface F. As the density of the net of points on F increases, the polyhedron P converges toward F. Here, if the image M on P in this projection is designated M_p, then

$$s\,(M_p) \rightarrow s\,(M) \quad \text{and} \quad \omega\,(M_p) \rightarrow \omega\,(M).$$

In the geodesic mapping the polyhedron P will correspond to the Euclidean polyhedron $\overline{P}$, between the points of which and the points $\overline{F}$ correspondence has been established by projection of the image $\overline{0}$ from the point $0 \cdot$ in geodesic representation, and when $P \rightarrow F$, $\overline{P} \rightarrow \overline{F}$,

$$\overline{s}\,(\overline{M}_p) \rightarrow \overline{s}\,(M), \quad \overline{\omega}\,(\overline{M}_p) \rightarrow \overline{\omega}\,(\overline{M}).$$

It follows from this that the lemma need only be proved for a case of polyhedra. Furthermore, since the area and curvature are additive set functions, the first of the inequalities in the lemma need only be established for a set lying on one face, and the second need only be established for the curvature at one apex.

Let us turn to the projective model of elliptic space in Euclidean space, $x_0 = 1$. Here the polyhedron

P is represented by the Euclidean polyhedron $\overline{P}$. Since the ratio of the change from distances in the Euclidean metric to distances in the elliptic metric in the bounded subset of space remains within positive limits, the ratio of the change from area in the Euclidean to area in elliptic metric is also within positive bounds. The first of the inequalities in the lemma follows from this.

The second inequality is established in the same way. Since the ratio of change from the angle of the planes in the Euclidean metric to the angle between them in the elliptic metric is contained within positive limits, the ratio of change from curvature of the apex in the Euclidean metric to curvature in the elliptic metric is also contained within positive limits. QED.

Using the Aleksandrov theorems mentioned above, it follows from lemma 2 that:

1. Convex surfaces of specific curvature bounded from above in elliptic space do not contain either rib or conical points.

2. Convex surfaces of specific curvature bounded from below are strictly convex, i.e., they do not contain any rectilinear segments.

4. PROOF OF THE REGULARITY OF CONVEX SURFACES WITH A REGULAR METRIC IN ELLIPTIC SPACE.

We will say that the metric of the surface is regular (differentiable k times) when it is possible to introduce the coordinate network u, v on the surface in such a way that the coefficients E, F and G of

the linear element of the surface, as functions of the coordinates u, v, are regular (are k times differentiable functions).

The regularity of the surface implies the regularity of its metric. That is, if the surface is differentiable k times, its metric is differentiable at least k — 1 times. Generally speaking, the opposite is false, the surface may have a regular, even analytic metric and not be regular. Nevertheless, the following theorem is valid:

Theorem 1. If a convex surface F in elliptic space has a regular metric (differentiable k times, k ⩾ 5) and Gaussian curvature of the surface greater than the curvature of space, the surface itself is regular (at least k — 1 times differentiable). Moreover, if the metric of the surface is analytic, the surface is analytic.

Proof. In a fairly small neighborhood G of any point x on the surface F, its Gaussian curvature K_i satisfies the inequality

$$1 < a \leqslant K_i \leqslant b.$$

Since the intrinsic curvature of the surface F is different from the extrinsic curvature by the magnitude of the area

$$\omega\,(M) = \int\int_M K_i ds - s\,(M),$$

the specific extrinsic curvature of the surface at G is contained within positive limits

$$a - 1 \leqslant \frac{\omega \, (M)}{s \, (M)} \leqslant b - 1.$$

It follows from this (para. 3) that the surface F is smooth and strictly convex.

Let us prove the regularity of the surface F in the neighborhood of the arbitrary point x. Let us first consider a case of an analytic metric.

Let the plane α cut from the surface F a small cap ω containing the point x which is small enough that any plane β intersecting F in ω also cuts off a cap. It is possible to cut off this cap ω because of the smoothness and strict convexity of the surface F.

Since the edge γ of the cap ω has a non-negative swerve in any arc, it can always be approximated by an analytic curve γ^* with a positive geodesic curvature. This curve bounds a certain region in ω, which we will call ω^*.

The two copies of the surface ω^* can be "joined" to form a closed manifold Ω by identifying the points on the boundaries which correspond isometrically. This manifold will have an analytical metric everywhere, except for the joining line. The copies need not be joined directly, however, but through a narrow regular zone, regular in the sense of the intrinsic metric passing at ω^*. Since the geodesic curvature of the edge of ω^* is positive, this zone can be selected in such a way that its Gaussian curvature is always greater than one (the curvature of space). I have described the operation of forming a manifold Ω from the two copies ω^* and a regular zone in greater detail in my publication [4].

In this publication, it is proved that in elliptic space there is a regular closed convex surface isometric to Ω. Let us designate the region corresponding to ω^* as $\widetilde{\omega}^*$.

It follows from the theorem of uniqueness of closed convex surfaces (Chapter VI) that as the width of the zone decreases to zero, any $\gamma^* \rightarrow \gamma$, Ω converges towards the closed convex surface which is formed by ω and its mirror image on the plane at the edge. That is to say, under these conditions

$$\widetilde{\omega}^* \rightarrow \omega.$$

Let us shift the plane α towards the cap into the position α', and designate the caps cut off from the surfaces ω and ω^*, by the plane α', as ω' and $\widetilde{\omega}'$, respectively.

Let $\widetilde{G}$ be a closed region of $\widetilde{\omega}'$ separated from the plane α' by a distance greater than $h_0 > 0$. An evaluation can be given in this region for normal curvatures of $\widetilde{\omega}'$ in terms of the internal metric of the surface F.

Estimation of normal curvatures is tantamount to estimation of the first and second derivatives of the coordinates x_i of a point on the surface. And this enables us, using the deformation equation, which is an equation of the elliptic type for x_i, to estimate the deriviates x_i of all orders. (The method of obtaining these estimates is described in [6], Appendix 2).

The existence of intrinsic estimates for all derivatives of x_i suggests that in the region $G = \lim \widetilde{G}$,

the surface F is infinitely differentiable. And by virtue of the ellipticality of the deformation equation, it follows that the coordinates x_i as functions of u, v are analytic, i.e., that the surface F is analytical.

Now let the metric of the surface F be differentiable k times. Let us approximate the metric F at ω by an analytic metric, and the curve γ^* by an analytic curve. Here we obtain an analytical manifold with an internally convex edge. According to Aleksandrov's theorem, this manifold is isometric to a cap. According to what we have proved, it is analytic. The possibility of obtaining the a priori estimates for derivatives of the coordinates of a point on this point of the kth order suggests that the limiting cap which by virtue of the uniqueness of convex caps is the region on F will be differentiable k — 1 times. And what is more, k — 1 derivatives of the coordinates of the points satisfy the Holder condition with the exponent 1.

The theorem is proved.

As a corollary to Theorem 1 and A. D. Aleksandrov's theorems on the realization of manifolds of curvature not less than K in elliptic space of curvature K we obtain Theorems 2 and 3.

Theorem 2. A closed two–dimensional manifold of curvature greater than K with a regular metric is isometric to a regular closed convex surface in elliptic space of curvature K. If the metric of the manifold is differentiable k times, $k \geqslant 5$, the surface is differentiable k — 1 times. If the metric is analytic, the surface is analytic.

Theorem 3. Each point in the two-dimensional manifold of curvature greater than K with a regular metric has a neighborhood isometric to a regular convex cap in elliptic space of curvature K. And if the metric of the manifold is differentiable $k \geqslant 5$ times, the cap is differentiable $k - 1$ times. If the metric of the manifold is analytic, then the cap is analytic.

In conclusion we should point out that the theorem of the regularity of a convex surface with a regular metric enables us to apply the synthetic methods of Aleksandrov's theory of general convex surfaces, in particular, the theorem of "gluing", as in Euclidean space, to the solution of various problems in the classical theory of surfaces, which usually deals with fairly regular objects.

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